IRREGULARITIES OF SPECIAL $\mathcal{C}$-PAIRS

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ABSTRACT. This paper studies irregularity-type invariants of special $\mathcal{C}$-pairs, or "geometric orbifolds" in the sense of Campana. Under mild assumptions on the singularities, we show that the augmented irregularity of a $\mathcal{C}$-pair (X, D) is bounded by its dimension. This generalizes earlier results of Campana, and strengthens known results even in the classic case where X is a projective manifold and $D = 0$. The proof builds on new extension results for adapted forms, analysis of foliations on Albanese varieties, and constructions of Bogomolov sheaves using strict wedge subspaces of adapted forms.

CONTENTS

1. Introduction 1
2. Notation and known results 3
3. Rational maps induced by generically generated sheaves 7
4. Extension of adapted reflexive differentials, Proof of Theorem 1.6 10
5. Bogomolov sheaves and linear systems in reflexive G-sheaves 14
6. Invariant Bogomolov sheaves defined by strict wedge subspaces 17
7. Bounds on invariants, Proof of Theorem 1.3 and Corollary 1.7 21

Appendix 24
Appendix A. Extension of low degree differentials 24
References 27

1. INTRODUCTION

Special varieties were introduced in a series of influential papers by Campana, [Cam04, Cam11], as complex-projective manifolds where the classic Bogomolov-Sommese inequality is strict, or equivalently, as complex-projective manifolds that do not dominate a "geometric orbifold" or "$\mathcal{C}$-pair" of general type. It is conjectured that the notion of "specialness" characterizes "potential density", both in the arithmetic setting (where "density" refers to sets of rational points) and in the analytic setting (where "density" refers to entire curves).

This article studies *irregularities* of special manifolds and of mildly singular $\mathcal{C}$-pairs that appear in the minimal model program.

Invariants of special manifolds. The starting point is a fundamental observation of Campana: If X is a complex-projective manifold that is special, then the irregularity $q(X) := h^0(X, \Omega^1_X)$ is always bounded by the dimension of X, [Cam04, Sect. 5.2]. Using his results

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on the invariance of specialness under étale coverings, he concludes that the *augmented irregularity*,

$$\widetilde{q}(X) := \sup \left\{ q(\widetilde{X}) \;:\; \widetilde{X} \to X \text{ a finite étale cover} \right\},$$

is likewise bounded by $\dim X$.

Invariants of special pairs. Given that the natural objects of Campana's theory are "geometric orbifolds" or "$\mathcal{C}$-pairs", where adapted differentials take the role that ordinary differentials play for ordinary spaces, it is natural to ask for generalizations.

Conjecture 1.1 (Irregularities of special $\mathcal{C}$-pairs, [KR24a, Conjecture 6.17]). Let (X, D) be a $\mathcal{C}$-pair where the analytic variety X is compact and Kähler. If (X, D) is special, then its augmented irregularity is bounded by the dimension, $q^+(X, D) \le \dim X$.

For the convenience of the reader not familiar with the theory, we recall the definition of "augmented irregularity" in brief. The reference paper [KR24a] introduces and discusses all relevant notions in great detail.

Definition 1.2 (Irregularity, augmented irregularity, [KR24a])**.** *Let (X, D) be a compact $\mathcal{C}$-pair. If $\gamma : \widehat{X} \to X$ is any cover, we refer to the number*

$$q(X, D, \gamma) := h^0\left(\widehat{X}, \Omega^{[1]}_{(X,D,\gamma)}\right)$$

as the irregularity *of (X, D, γ). The number*

$$q^+(X, D) := \sup\bigl\{ q(X, D, \gamma) \;:\; \gamma \text{ a cover} \bigr\} \in \mathbb{N} \cup \{\infty\}$$

is the augmented irregularity *of the $\mathcal{C}$-pair (X, D).*

1.1. **Main result.** The main result of this paper answers Conjecture 1.1 in the positive, for all pairs that will typically appear in minimal model theory. We refer the reader to [KM98] for the definition of "divisorially log terminal" pairs.

Theorem 1.3 (Boundedness of augmented irregularity)**.** *Let (X, D) be a $\mathcal{C}$-pair that satisfies one of the following conditions.*

(1.3.1) The analytic variety X is compact Kähler and (X, D) is locally uniformizable.
(1.3.2) The analytic variety X is projective and (X, D) is divisorially log terminal (=dlt).

If (X, D) is special, then $q^+(X, D) \le \dim X$.

Remark 1.4 (Novelty of the result). Even in cases where X is a complex-projective manifold and $D = 0$, Theorem 1.3 is new and stronger than Campana's classic result, which considers étale coverings only.

Remark 1.5 (Earlier results on Albanese irregularities). Theorem 1.3 generalizes and strengthens earlier results, including [KR24b, Theorem 8.1], on the "augmented Albanese irregularity" $q^+_{\mathrm{Alb}}(X, D)$. The augmented Albanese irregularity is a variant of the augmented irregularity. It is geometrically meaningful, always bounded by $q^+(X, D)$, but hard to control and compute in practise.

The proof of Theorem 1.3 relies in part on the following extension theorem for adapted reflexive differentials, which might be of independent interest.

Theorem 1.6 (Extension of adapted forms on dlt pairs)**.** *Let (X, D) be an algebraic, quasi-projective $\mathcal{C}$-pair that is dlt. Then, adapted reflexive 1-forms on (X, D) extend to log resolutions of singularities.*

1.2. **Outline of the proof.** For the proof of the classic result on irregularities of special complex-projective manifolds, it suffices to show that the Albanese map of a special manifold is necessarily surjective. Aiming for the contrapositive, Campana studies manifolds whose Albanese is *not* surjective. Building on earlier of work of Kawamata, Kobayashi, and Ueno on positivity in sheaves of differentials on varieties of maximal Albanese dimension, he constructs a sheaf of differentials where equality in the Bogomolov-Sommese inequality is attained, showing that the underlying space cannot be special.

In our setting, where adapted differentials take the role that ordinary differentials play for ordinary spaces, there is in general no "adapted Albanese map". Even in special cases where a suitably-defined Albanese does exist, it is known that the classical equality between the irregularity and the dimension of the Albanese variety is not true in general [KR24b, Sect. 7.1].

We overcome this problem by considering the classic Albanese of suitable covers, where adapted differentials define a foliation. Though these will typically have Zariski dense leaves, we can leverage ideas from Catanese's work on generalized Castelnuovo-De Franchis theorems and "strict wedge subspaces of differentials", in order to obtain positivity results for the foliated variety that can be used in lieu of the classic arguments.

For sheaves of one-forms, these arguments work particularly well, and provide the following partial generalization of Campana's statement on the invariance of specialness under étale cover, [Cam04, Sect. 5.2] and [Cam11, Prop. 10.11].

Corollary 1.7 (Adapted one-forms on covers of spacial pairs)**.** *Let (X, D) be a projective C-pair that is dlt, let $\gamma : \widehat{X} \twoheadrightarrow X$ be any cover and let $\mathscr{L} \subseteq \Omega^{[1]}_{(X,D,\gamma)}$ be coherent of rank one. If (X, D) is special, then the C-Kodaira-Iitaka dimension of $\mathscr{L}$ is bounded by one: $\kappa_C(\mathscr{L}) < 1$.*

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2. Notation and known results

This paper works with complex spaces. With very few exceptions, we follow the notation of the standard reference texts [GR84, Dem12]. This section clarifies less commonly-used notation and recalls a few well-known results for later reference. A full introduction to the theory of C-pairs is, however, out of scope. We refer the reader to the reference [KR24a] for definitions and a very detailed introduction to all the material used here. For the reader's convenience, we include precise references to [KR24a] throughout the text, whenever a term of C-pair theory appears for the first time.

2.1. **Global assumptions and standard notation.** An *analytic variety* is a reduced, irreducible complex space. For clarity, we refer to holomorphic maps between analytic varieties as *morphisms* and reserve the word *map* for meromorphic mappings.

Definition 2.1 (Big and small sets)**.** *Let X be an analytic variety. An analytic subset $A \subsetneq X$ is called* small *if it has codimension two or more. An open set $U \subseteq X$ is called* big *if $X \setminus U$ is analytic and small.*

Definition 2.2 (q-morphisms)**.** *Quasi-finite morphisms between normal analytic varieties of equal dimension are called q-*morphisms.

Following the literature, we use square brackets to denote reflexive tensor operations.

Notation 2.3 (Reflexive tensor operations). Let X be a normal analytic variety and let $\mathscr{L}$ be a torsion free coherent sheaf of $\mathscr{O}_X$-modules. Write

$$\mathscr{L}^{[\otimes n]} := (\mathscr{L}^{\otimes n})^{**}, \qquad \wedge^{[n]}\mathscr{L} := (\wedge^n \mathscr{L})^{**},$$
$$\operatorname{Sym}^{[n]}\mathscr{L} := (\operatorname{Sym}^n \mathscr{L})^{**}, \qquad \det \mathscr{L} := (\wedge^{\operatorname{rank}\mathscr{L}} \mathscr{L})^{**}.$$

If $\varphi : X \to Y$ is a morphism and $\mathscr{F}$ a coherent sheaf on Y, write $\varphi^{[*]}\mathscr{F} := (\varphi^*\mathscr{F})^{**}$.

If X is any analytic variety, we denote the sheaf of Kähler differentials by $\Omega^\bullet_X$. We recall the notation for reflexive logarithmic differentials.

Notation 2.4 (NC locus). Let X be a normal analytic variety and let D be a Weil $\mathbb{Q}$-divisor on X. Write $(X, D)_{\mathrm{reg}} \subseteq X$ for the maximal open set where X is smooth and D has normal crossing support.

Notation 2.5 (Differentials with logarithmic poles). Let X be a normal analytic variety and let D be a Weil $\mathbb{Q}$-divisor on X.

(2.5.1) If X is smooth and D has nc support, we will often write $\Omega^p_X(\log D)$ to denote the sheaves of Kähler differentials with logarithmic poles along $\operatorname{supp} D$.

(2.5.2) Denote the inclusion of the nc locus by $\iota : (X, D)_{\mathrm{reg}} \hookrightarrow X$ and write

$$\Omega^{[p]}_X(\log D) = \iota_* \Omega^p_{(X,D)_{\mathrm{reg}}}\big(\log D|_{(X,D)_{\mathrm{reg}}}\big).$$

(2.5.3) If $\pi : Y \to X$ is any morphism from a smooth variety Y where $\pi^{-1} \operatorname{supp} D$ is of pure codimension one and has normal crossings, write $\Omega^p_Y(\log \pi^* D)$ to denote the sheaves of Kähler differentials with logarithmic poles along the set $\pi^{-1} \operatorname{supp} D$.

Remark 2.6. The sheaf $\Omega^{[p]}_X(\log D)$ in (2.5.2) is reflexive, in particular coherent.

For later reference, we note the following lemma on the behaviour of pull-back for differentials under product maps. The proof is elementary and left to the reader.

Lemma 2.7 (Pull-back of differentials under product maps)**.** *Let $(\varphi_i : X \dashrightarrow Y_i)_{i=1,\dots,a}$ be a finite number of rational maps between complex manifolds and let*

$$\varphi := \varphi_1 \times \cdots \times \varphi_a : X \dashrightarrow Y_1 \times \cdots \times Y_a$$

be the associated product map. Then, there exists a dense open subset $X^\circ \subseteq X$ where all maps are well-defined and where the following subsheaves of $\Omega^1_{X^\circ}$ agree,

$$\operatorname{img} d(\varphi|_{X^\circ}) = \sum_{i=1}^{a} \operatorname{img} d(\varphi_i|_{X^\circ}) \subseteq \Omega^1_{X^\circ}. \qquad \square$$

2.2. ***C*-pairs and adapted morphisms.** The key notion of the present paper is the C-pair. We recall the definition in brief and refer to the reference paper [KR24a] for details.

Definition 2.8 (C-pairs, [KR24a, Sect. 2.5])**.** *A C-pair is a pair (X, D) where X is a normal analytic variety and D a Weil $\mathbb{Q}$-divisor D is of the form*

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i,$$

with $m_i \in \mathbb{N}^{\geq 2} \cup \{\infty\}$ and $\frac{\infty - 1}{\infty} = 1$. If (X, D) is a C-pair, it will sometimes be convenient to consider the following Weil $\mathbb{Q}$-divisor

$$D_{\mathrm{orb}} := \sum_{i \,|\, m_i < \infty} \frac{1}{m_i} \cdot D_i \in \mathbb{Q}\operatorname{Div}(X).$$

Definition 2.9 (Adapted morphism, [KR24a, Sect. 2.5])**.** *Given a C-pair (X, D), a q-morphism $\gamma : \widehat{X} \to X$ is called* adapted for (X, D) *if $\gamma^* D_{\mathrm{orb}}$ is integral. It is called* strongly adapted for (X, D) *if $\gamma^* D_{\mathrm{orb}}$ is reduced.*

2.3. **Linear systems in reflexive sheaves.** If X is a compact manifold, $\mathscr{L} \in \operatorname{Pic}(X)$ a line bundle and $L \subseteq H^0(X, \mathscr{L})$ a non-trivial space of sections, complex geometry frequently considers the meromorphic map $\varphi_{L,\mathscr{L}} : X \dashrightarrow \mathbb{P}(L)$, given at general points by $x \mapsto \ker(\sigma \mapsto \sigma(x))$. Parts of this paper discuss analogous constructions in cases where X is potentially singular and $\mathscr{L}$ is reflexive of rank one. To avoid any confusion, we clarify the setup in detail.

Notation 2.10 (Projectivized linear spaces). If L is a linear space, write $\mathbb{P}(L)$ for the space of hyperplanes in L, or equivalently, for the space of one-dimensional subspaces in L^*.

Remark 2.11 (Subspaces and projections). With Notation 2.10, an inclusion $L_1 \hookrightarrow L_2$ of linear spaces therefore corresponds a linear projection $\mathbb{P}(L_2) \dashrightarrow \mathbb{P}(L_1)$, given at general points by $H \mapsto H \cap L_1$.

Lemma 2.12 (Spaces of sections in torsion free sheaves)**.** *If X is a compact, normal analytic variety, $\mathscr{L}$ a torsion free, rank-one sheaf of $\mathscr{O}_X$-modules and $L \subseteq H^0(X, \mathscr{L})$ a non-trivial space of sections, there exists a meromorphic map $\varphi_{L,\mathscr{L}} : X \dashrightarrow \mathbb{P}(L)$, given at general points by $x \mapsto \ker(\sigma \mapsto \sigma(x))$.*

Proof. Recall from [Ros68, Thm. 3.5] that there exists a bimeromorphic modification, say $\pi : \widetilde{X} \to X$, where $\mathscr{F} := (\pi^*\mathscr{L})/\mathrm{tor}$ is invertible. Let $F \subseteq H^0(\widetilde{X}, \mathscr{F})$ be the image of L under the natural inclusion

$$\pi^*/\mathrm{tor} : H^0(X, \mathscr{L}) \hookrightarrow H^0(\widetilde{X}, \mathscr{F}),$$

set $\varphi_{L,\mathscr{L}} := \varphi_{F,\mathscr{F}} \circ \pi^{-1}$ and observe that the map does not depend on the choice of the modification. □

Remark 2.13. The reader coming from algebraic geometry might wonder why Lemma 2.12 requires any proof at all: in contrast to the algebraic setting, holomorphic morphisms defined on a Zariski dense open do not in general extend to meromorphic mapping on the full space.

Notation 2.14 (Meromorphic maps induced by linear systems in torsion free sheaves). Throughout the present paper, we use the notation $\varphi_{L,\mathscr{L}}$ to denote the meromorphic map of Lemma 2.12. In cases where $L = H^0(X, \mathscr{L})$, we write $\varphi_{\mathscr{L}}$ instead of $\varphi_{H^0(X,\mathscr{L}),\mathscr{L}}$.

2.4. **SNC morphisms.** While snc pairs are the logarithmic analogues of smooth spaces, snc morphisms are the analogues of smooth maps. We recall the main properties for the reader's convenience and refer to [GKKP11, Sect. 2.B] for a full discussion and for references to the literature. While [GKKP11] works with algebraic varieties, the results mentioned here carry over to the analytic setting without change.

Notation 2.15 (Intersection of boundary components). Let (X, D) be a pair,

$$D = \alpha_1 \cdot D_1 + \ldots + \alpha_n \cdot D_n.$$

If $I \subseteq \{1, \ldots, n\}$ is not empty, write $D_I := \cap_{i \in I} D_i$ for the (potentially non-reduced) intersection of complex spaces. If I is empty, set $D_I := X$.

Reminder 2.16 (Description of snc pairs, [GKKP11, Rem. 2.8]). In the setup of Notation 2.15, the pair (X, D) is snc if and only if the following holds for every index set $I \subseteq \{1, \ldots, n\}$ with $D_I \neq \emptyset$.

(2.16.1) The intersection D_I is smooth.
(2.16.2) The codimension equals $\operatorname{codim}_X D_I = |I|$.

Definition 2.17 (Snc morphism, [GKKP11, Def. 2.9])**.** *Let (X, D) be an snc pair and let $\varphi : X \twoheadrightarrow T$ be a surjective morphism to a complex manifold. Call φ* an snc morphism of (X, D) *if the following holds for every index set $I \subseteq \{1, \ldots, n\}$ with $D_I \neq \emptyset$.*

(2.17.1) The restricted morphism $\varphi|_{D_I} : D_I \to T$ is smooth.

(2.17.2) *The restricted morphism $\varphi|_{D_I}$ has relative dimension* $\dim X - \dim T - |I|$.

Reminder 2.18 (All morphisms are generically snc). If (X, D) is an snc pair and $\varphi : X \to T$ is a surjective morphism to a complex manifold, then there exists a dense, Zariski open subset of T over which φ is an snc morphism.

Reminder 2.19 (Fibre bundle structure of proper morphisms). Let (X, D) be a logarithmic snc pair and let $\varphi : X \twoheadrightarrow T$ be a proper snc morphism of (X, D). Then, $\varphi : X \setminus D \twoheadrightarrow T$ is a differentiable fibre bundle.

Reminder 2.20 (Fibers and relative differentials of snc morphisms). Let (X, D) be a logarithmic snc pair and let $\varphi : X \twoheadrightarrow T$ be an snc morphism of (X, D). Then, there exists a natural exact sequence of locally free sheaves,

$$0 \longrightarrow \varphi^*\Omega^1_T \xhookrightarrow{d\varphi} \Omega^1_X(\log D) \xrightarrow{q} \underbrace{\Omega^1_X(\log D)\big/\varphi^*\Omega^1_Y}_{=:\Omega^1_{X/T}(\log D)} \longrightarrow 0.$$

If $t \in T$ is any point with fibre $X_t := \varphi^{-1}(t)$ and inclusion $\iota_t : X_t \to X$, then X_t is smooth and $D_t := \iota_t^* D$ is logarithmic and snc. There exists a natural identification between restrictions rendering the following diagram commutative,

$$\begin{tikzcd}
\iota^*\Omega^1_X(\log D) \arrow[r, two heads, "\iota_t^* q"] \arrow[d, equal] & \iota_t^*\Omega^1_{X/T}(\log D) \arrow[d, hook, two heads, "\text{ident.}"] \\
\iota^*\Omega^1_X(\log D) \arrow[r, two heads, "d\iota_t"'] & \Omega^1_{X_t}(\log D_t)
\end{tikzcd}$$

2.5. **Forms on fibre spaces.** We recall (and slightly generalize) a fundamental fact of Kähler geometry: If $\varphi : X \to T$ is a fibration and σ a closed 1-form on X that vanishes on one fibre, then σ comes from T.

Proposition 2.21. *Let (X, D) be a logarithmic snc pair and let $\varphi : X \twoheadrightarrow T$ be an snc morphism of (X, D). Assume that X is Kähler and that φ is proper. If $\sigma \in H^0(X, \Omega^1_X(\log D))$ is closed, then the following statements are equivalent.*

(2.21.1) *There exists one point $t \in T$ with fibre $X_t := \varphi^{-1}(t)$ and inclusion $\iota_t : X_t \to X$ such that the restriction of σ to X_t vanishes,*

$$d\iota_t\sigma = 0 \in H^0\big(X_t,\ \Omega^1_{X_t}(\log D_t)\big).$$

(2.21.2) *Locally on X, the form σ comes from downstairs: $\sigma \in H^0(X, \varphi^*\Omega^1_T)$.*

Proof. The direction (2.21.2) $\Rightarrow$ (2.21.1) is trivial. To prove the converse, assume that a closed form σ and a point $t \in T$ with the properties of (2.21.1) are given. We need to show that

$$\sigma \in H^0(X, \varphi^*\Omega^1_T) \overset{\text{Reminder 2.20}}{\Leftrightarrow} d\iota_s\sigma = 0 \in H^0\big(X_s,\ \Omega^1_{X_s}(\log D_s)\big), \text{ for every } s \in T.$$

Assume that a point $s \in T$ is given. Since (X_s, D_s) is snc and X_s is compact and Kähler, recall that the image of the natural integration map,

$$\pi_1(X_s \setminus D_s) \to H^0\big(X_s,\ \Omega^1_{X_s}(\log D_s)\big)^*, \quad \gamma \mapsto \Big(\tau \mapsto \int_\gamma \tau\Big)$$

spans the vector space $H^0\big(X_s,\ \Omega^1_{X_s}(\log D_s)\big)^*$, see [NW14, Thm. 4.5.4]. To prove that $d\iota_s\sigma$ vanishes, it will therefore suffice to prove that

$$\int_{\gamma_s} d\iota_s\sigma = 0, \quad \text{for every loop } \gamma_s \text{ in } X_s \setminus D_s.$$

Assume that a loop γ_s in $X_s \setminus D_s$ is given and recall from Reminder 2.19 that $\varphi : X \setminus D \twoheadrightarrow T$ is a differentiable fibre bundle over a path connected base. The homotopy exact sequence therefore implies that the loop γ_s is homotopy equivalent within $X \setminus D$ to a loop γ_t in $X_t \setminus D_t$. We have

$$\int_{\gamma_s} d\iota_s \sigma = \int_{\iota_s \circ \gamma_s} \sigma \overset{(*)}{=} \int_{\iota_t \circ \gamma_t} \sigma = \int_{\gamma_t} d\iota_t \sigma \overset{(2.21.1)}{=} \int_{\gamma_t} 0 = 0$$

where $(*)$ follows from homotopy equivalence of γ_s and γ_t, and closedness of the holomorphic form σ. $\square$

2.6. **Foliations defined by meromorphic maps.** The conclusion of Proposition 2.21 is frequently summarized by saying that "the logarithmic form σ annihilates the foliation defined by φ". To avoid confusion, we define this terminology explicitly.

Definition 2.22 (Foliation defined by meromorphic maps)**.** *If $\varphi : X \dashrightarrow Y$ is a meromorphic map between complex manifolds, let $\iota : X^\circ \hookrightarrow X$ be the inclusion of the maximal open set where φ is well-defined. We refer to*

$$\iota_* \ker\Big(T(\varphi|_{X^\circ}) : \mathscr{T}_{X^\circ} \to (\varphi|_{X^\circ})^* \mathscr{T}_Y\Big) \subseteq \iota_* \mathscr{T}_{X^\circ} = \mathscr{T}_X \tag{2.22.1}$$

as the foliation defined by φ*. Observe that this sheaf is coherent, saturated as a subsheaf of $\mathscr{T}_X$ and hence reflexive. It is closed under the Lie bracket.*

Notation 2.23 (Foliation defined by meromorphic maps). Assume the setting of Definition 2.22. If no confusion is likely to arise, we write $\ker(T\varphi) \subseteq \mathscr{T}_X$ for the foliation defined by φ, as a shorthand for the more cumbersome expression (2.22.1).

Definition 2.24 (Logarithmic forms annihilating foliations)**.** *If (X, D) is an snc pair, if $\mathscr{F} \subseteq \mathscr{T}_X$ is a foliation and $\sigma \in H^0\big(X,\, \Omega^1_X(\log D)\big)$ a logarithmic form, we say that "σ annihilates $\mathscr{F}$" if the natural sheaf morphism*

$$\sigma : \mathscr{F} \to \mathscr{O}_X(D_{\mathrm{red}})$$

is constantly zero.

3. Rational maps induced by generically generated sheaves

To prepare for the discussion of "Bogomolov sheaves" defined by strict wedge subspaces in Section 6, this section considers sheaves $\mathscr{E}$, subspaces $E \subseteq H^0\big(X,\, \mathscr{E}\big)$ and rational maps coming from the induced linear systems $\wedge^{\operatorname{rank} \mathscr{E}} E \to H^0\big(X,\, \det \mathscr{E}\big)$.

Definition 3.1 (Sheaves generically generated by spaces of sections)**.** *Let X be a compact and normal analytic variety, and let $\mathscr{E}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Assume that $\mathscr{E}$ is not a torsion sheaf, and let $E \subseteq H^0\big(X,\, \mathscr{E}\big)$ be a linear subspace. Call $\mathscr{E}$* generically generated by sections in E *if the following two equivalent conditions hold.*

(3.1.1) The natural map $E \otimes \mathscr{O}_X \to \mathscr{E}$ is generically surjective.
(3.1.2) The natural map $\lambda_E : \wedge^{\operatorname{rank} \mathscr{E}} E \to H^0\big(X,\, \det \mathscr{E}\big)$ is non-trivial.

The following construction is the basis for all that follows in this section.

Construction 3.2 (Projection to linear systems induced by spaces of sections). In the setup of Definition 3.1, assume that $\mathscr{E}$ is generically generated by sections in E. Observe that $\det \mathscr{E}$ is reflexive with a non-trivial space of sections. Following the notation of Section 2.3, denote the associated rational map by

$$\varphi_{\det \mathscr{E}} : X \dashrightarrow \mathbb{P}\Big(H^0\big(X,\, \det \mathscr{E}\big)\Big).$$

As remarked in 2.11, the inclusion $\operatorname{img} \lambda_E \subseteq H^0\big(X,\, \det \mathscr{E}\big)$ of linear spaces induces a rational projection map between the projectivizations,

$$\mathbb{P}\Big(H^0\big(X,\, \det \mathscr{E}\big)\Big) \dashrightarrow \mathbb{P}\Big(\operatorname{img} \lambda_E\Big).$$

Given that elements of $\operatorname{img} \lambda_E$ do not vanish at general points of X, the image of $\varphi_{\det \mathscr{E}}$ is not contained in the indeterminacy of the projection, and we obtain a composed rational map $\eta_E := \varphi_{\operatorname{img} \lambda_E, \det \mathscr{E}}$ as follows,

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{\det \mathscr{E}}"'] \arrow[rr, dashed, bend left, "\eta_E"] & \mathbb{P}\Big(H^0\big(X,\, \det \mathscr{E}\big)\Big) \arrow[r, dashed, two heads, "\text{projection}"'] & \mathbb{P}\Big(\operatorname{img} \lambda_E\Big). \end{tikzcd}$$

Notation 3.3 (Projection to linear systems induced by spaces of sections). We will use the notation η_E of Construction 3.2 throughout the present paper, whenever we discuss sheaves generically generated by spaces of sections.

Remark 3.4 (Sheaves of differentials generically generated by spaces of sections). Consider the setup of Construction 3.2 in case X is a Kähler manifold, D an snc divisor on X and $\mathscr{E} \subseteq \Omega^1_X(\log D)$ a sheaf of logarithmic differentials. Then, there exists a sequence of inequalities,

$$\dim \operatorname{img} \eta_E \le \dim \operatorname{img} \varphi_{\det \mathscr{E}} \le \kappa(\det \mathscr{E}) \le \operatorname{rank} \mathscr{E},$$

where the last inequality is given by the classic vanishing theorem of Bogomolov-Sommese[1]. Recall that $\det \mathscr{E}$ is called a *Bogomolov sheaf* if the equality $\kappa(\det \mathscr{E}) = \operatorname{rank} \mathscr{E}$ holds.

3.1. **Functoriality.** The following proposition asserts that projection to linear systems induced by spaces of sections is functorial in inclusions of sheaves generically generated by spaces of sections. The proof is tedious, but elementary and certainly not surprising. We include full details for completeness' sake.

Proposition 3.5 (Functoriality of $\eta_\bullet$ in inclusions)**.** *Let X be a compact and normal analytic variety and let $\mathscr{F} \subseteq \mathscr{E}$ be an inclusion of torsion free, coherent sheaves of $\mathscr{O}_X$-modules. Assume that $\mathscr{F}$ and $\mathscr{E}$ are generically generated by sections in $F \subseteq H^0\big(X,\, \mathscr{F}\big)$ and $E \subseteq H^0\big(X,\, \mathscr{E}\big)$, respectively. If $F \subseteq E$, then there exists a commutative diagram of composable rational maps,*

$$\begin{tikzcd} X \arrow[r, dashed, "\eta_E"'] \arrow[rr, dashed, bend left, "\eta_F"] & \mathbb{P}\Big(\operatorname{img} \lambda_E\Big) \arrow[r, dashed, two heads, "\exists \eta_{E,F}"'] & \mathbb{P}\Big(\operatorname{img} \lambda_F\Big), \end{tikzcd} \tag{3.5.1}$$

where $\eta_{E,F}$ is a linear projection.

Notation 3.6 (Functoriality of $\eta_\bullet$ in inclusions). We will use the notation $\eta_{E,F}$ of Proposition 3.5 throughout the paper, whenever we discuss inclusions of sheaves generically generated by spaces of sections.

Proof of Proposition 3.5. Choose a general point $x \in X$ and let $\tau_1, \dots, \tau_a \in E$ be a sequence of sections such that the classes of $\tau_1(x), \dots, \tau_a(x)$ form a basis of the quotient space $\mathscr{E}_x / \mathscr{F}_x$. We obtain a linear injection

$$\iota_\tau : H^0\big(X,\, \det \mathscr{F}\big) \hookrightarrow H^0\big(X,\, \det \mathscr{E}\big), \quad \sigma \mapsto \sigma \wedge \tau_1 \wedge \cdots \wedge \tau_a.$$

[1] See [EV92, Cor. 6.9] for the projective case and [LMN+25, Sect. 6] for a discussion of the vanishing theorem in the non-projective Kähler setting

Observing that the injection ι_τ restricts to a linear injection between the images of the λ-operators, we obtain a commutative diagram of linear injections,

$$\begin{tikzcd} \operatorname{img}\lambda_E \arrow[r, hook] & H^0\big(X,\, \det\mathscr{E}\big) \\ \operatorname{img}\lambda_F \arrow[r, hook] \arrow[u, hook, "\iota_\tau|_{\operatorname{img}\lambda_F}"] & H^0\big(X,\, \det\mathscr{F}\big), \arrow[u, hook, "\iota_\tau"'] \end{tikzcd}$$

and hence a diagram of composable rational maps between the projectivizations,

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{\det\mathscr{E}}"'] \arrow[rr, dashed, bend left, "\eta_E"] \arrow[d, equal] & \mathbb{P}\Big(H^0\big(X,\, \det\mathscr{E}\big)\Big) \arrow[r, dashed, two heads, "\text{projection}"'] \arrow[d, dashed, two heads, "\mathbb{P}(\iota_\tau)"'] & \mathbb{P}\Big(\operatorname{img}\lambda_E\Big) \arrow[d, dashed, two heads, "\mathbb{P}(\iota_\tau|_{\operatorname{img}\lambda_F})"] \\ X \arrow[r, dashed, "\varphi_{\det\mathscr{F}}"] \arrow[rr, dashed, bend right, "\eta_F"'] & \mathbb{P}\Big(H^0\big(X,\, \det\mathscr{F}\big)\Big) \arrow[r, dashed, two heads, "\text{projection}"] & \mathbb{P}\Big(\operatorname{img}\lambda_F\Big), \end{tikzcd} \tag{3.7.1}$$

whose right square commutes by construction. We are done once we prove that the left square of (3.7.1) also commutes. To this end, let $x \in X$ be a general point. Identifying points of projective spaces with codimension-one subspaces of the underlying vector spaces, we have

$$\begin{aligned} \varphi_{\det\mathscr{E}}(x) &= \big\{\mu \in H^0\big(X,\, \det\mathscr{E}\big) \;:\; \mu(x) = 0\big\} \\ \varphi_{\det\mathscr{F}}(x) &= \big\{\sigma \in H^0\big(X,\, \det\mathscr{F}\big) \;:\; \sigma(x) = 0\big\} \end{aligned}$$

and then

$$\begin{aligned} \mathbb{P}(\iota_\tau)\big(\varphi_{\det\mathscr{E}}(x)\big) &= \iota_\tau^{-1}\big\{\mu \in H^0\big(X,\, \det\mathscr{E}\big) \;:\; \mu(x) = 0\big\} \\ &= \big\{\sigma \in H^0\big(X,\, \det\mathscr{F}\big) \;:\; (\sigma \wedge \tau_1 \wedge \cdots \wedge \tau_a)(x) = 0\big\}. \end{aligned}$$

But since x is general and since forming a basis is a Zariski open property, the choice of $\tau_\bullet$ guarantees that

$$\sigma(x) = 0 \Leftrightarrow (\sigma \wedge \tau_1 \wedge \cdots \wedge \tau_a)(x) = 0, \quad \text{for every } \sigma \in H^0\big(X,\, \det\mathscr{F}\big),$$

which is the desired commutativity statement. □

For later use in Section 6, we remark that Proposition 3.5 applies in a setting where the larger sheaf is obtained as a finite sum of subsheaves.

Corollary 3.8 (Sums of sheaves generically generated by spaces of sections)**.** *Let X be a projective manifold and let $\mathscr{F}_1, \dots, \mathscr{F}_a \subseteq \mathscr{E}$ be inclusions of torsion free, coherent sheaves of $\mathscr{O}_X$-modules. Assume that the $\mathscr{F}_\bullet$ are generically generated by sections in $F_\bullet \subseteq H^0\big(X,\, \mathscr{F}_\bullet\big)$ and set*

$$\mathscr{F} := \sum_i \mathscr{F}_i \subseteq \mathscr{E} \qquad F := \sum_i F_i \subseteq H^0\big(X,\, \mathscr{E}\big).$$

Then, $\mathscr{F}$ is generically generated by sections in F and there exists a commutative diagram of composable rational maps,

$$\begin{tikzcd} X \arrow[r, dashed, "\eta_F"'] \arrow[rr, dashed, bend left, "\eta_{F_1}\times\cdots\times\eta_{F_a}"] & \mathbb{P}\Big(\operatorname{img}\lambda_F\Big) \arrow[r, dashed, "\exists\eta"'] & \mathbb{P}\Big(\operatorname{img}\lambda_{F_1}\Big) \times\cdots\times \mathbb{P}\Big(\operatorname{img}\lambda_{F_a}\Big). \end{tikzcd} \tag{3.8.1}$$

In particular, we have

$$\dim\operatorname{img}\eta_F \geq \dim\operatorname{img}\eta_{F_1}\times\cdots\times\eta_{F_a}. \tag{3.8.2}$$

Proof. The assertion that $\mathscr{F}$ is generically generated by sections in F is clear. Apply Proposition 3.5 to the subsheaves $\mathscr{F}_\bullet \subset \mathscr{F}$ and take $\eta := \eta_{F,F_1} \times \cdots \times \eta_{F,F_a}$. Inequality (3.8.2) follows from commutativity of (3.8.1), which guarantees that the image of η_F dominates the image of $\eta_{F_1} \times \cdots \times \eta_{F_a}$. □

4. Extension of adapted reflexive differentials, Proof of Theorem 1.6

Theorem 1.6 asserts that reflexive 1-forms on dlt $\mathcal{C}$-pairs extend to log resolutions of singularities. Sections 4.1 and 4.2 make this statement precise and compare the notion to the "pull-back" discussed in [KR24a, Sect. 5]. The subsequent Section 4.3 provides elementary extendability criteria, which are then applied in Section 4.4 to prove Theorem 1.6.

4.1. **Definition.** The extension problem for adapted reflexive differentials considers a cover $\widehat{X}$ of a $\mathcal{C}$-pair, a resolution $\widetilde{X}$ of the singularities and asks if every adapted reflexive differential comes from a logarithmic differential on $\widetilde{X}$. We consider the following setting throughout the present section.

Setting 4.1 (Extension of adapted reflexive differentials). Given a $\mathcal{C}$-pair (X, D), consider sequences of morphisms of the following form,

$$\widetilde{X} \xrightarrow{\pi,\text{ log resolution of } \left(\widehat{X},\, \gamma^* \lfloor D \rfloor\right)} \widehat{X} \xrightarrow{\gamma,\ q\text{-morphism}} X. \tag{4.1.1}$$

Write

$$\widehat{X}^\circ := \widehat{X}_{\text{reg}} \cap \gamma^{-1}(X_{\text{reg}} \setminus \operatorname{supp} D)$$

and observe that $\widehat{X}^\circ$ is a non-empty, Zariski open subset of $\widehat{X}$.

Remark 4.2 (Sheaves of reflexive log differentials). Maintain Setting 4.1. If $p \in \mathbb{N}$ is any number, we consider the natural sheaves of reflexive log differentials,

$$\Omega^{[p]}_{(X,D,\gamma)} \subseteq \Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor) \tag{4.2.1}$$

$$\pi_* \Omega^p_{\widetilde{X}}(\log \pi^* \gamma^* \lfloor D \rfloor) \subseteq \Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor). \tag{4.2.2}$$

We refer the reader to [KR24a, Sect. 4.2] where Inclusion (4.2.1) is discussed at length. Inclusion (4.2.2) exists because the sheaf on the left is torsion free and agrees with the reflexive sheaf on the right on the big open set where the resolution map π is isomorphic.

Definition 4.3 (Extension of adapted reflexive differentials)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. Say that* adapted reflexive p-forms on (X, D) extend to log resolutions of singularities *if for every sequence of the form* (4.1.1)*, we have*

$$\Omega^{[p]}_{(X,D,\gamma)} \subseteq \pi_* \Omega^p_{\widetilde{X}}(\log \pi^* \gamma^* \lfloor D \rfloor). \tag{4.3.1}$$

Remark 4.4 (Sheaves in Definition 4.3). Recall from Remark 4.2 that the sheaves on the left and right are subsheaves of $\Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor)$. Inclusion (4.3.1) is therefore meaningful.

The word "extension" in Definition 4.3 is justified by the following remark.

Remark 4.5 (Extension of globally defined adapted reflexive differentials). Assume that Inclusion (4.3.1) of Definition 4.3 holds. If

$$\widehat{\sigma} \in H^0\big(\widehat{X},\ \Omega^{[p]}_{(X,D,\gamma)}\big) \subseteq H^0\big(\widehat{X},\ \Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor)\big),$$

is any adapted reflexive p-form, then there exists a logarithmic differential

$$\widetilde{\sigma} \in H^0\big(\widetilde{X},\ \Omega^p_{\widetilde{X}}(\log \pi^* \gamma^* \lfloor D \rfloor)\big)$$

that agrees over $(\widehat{X}, \gamma^* \lfloor D \rfloor)_{\text{snc}}$ with the pull-back of $\widehat{\sigma}$.

Proposition 4.6 (Independence of the resolution)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. Then, the following statements are equivalent.*

(4.6.1) Inclusion (4.3.1) holds for every sequence of the form (4.1.1),
(4.6.2) Inclusion (4.3.1) holds for one sequence of the form (4.1.1).

Proof. Given that any two log resolutions are dominated by a common third, the subsheaf $\pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) \subseteq \Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor)$ does not depend on π. □

4.2. **Relation to the literature.** For locally uniformizable pairs[2], Section 5 of the reference paper [KR24a] discusses "pull-back", a concept closely related to the "extension" of Definition 4.3. The definition of "pull-back" replaces sequences of the form (4.1.1) by sequences where π is an arbitrary morphism from a smooth space, and replaces Inclusion (4.3.1) by the existence of a pull-back morphism "$d_{\mathcal{C}}\pi$" that generalizes the standard pull-back of Kähler differentials. The following proposition relates these notions for later reference.

Proposition 4.7 (Extension as a pull-back property)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. Adapted reflexive p-forms on (X, D) extend to log resolutions of singularities if and only if for every sequence of the form (4.1.1), there exists an injective sheaf morphism*

$$d_{\mathcal{C}}\pi : \pi^{[*]}\Omega^{[p]}_{(X,D,\gamma)} \hookrightarrow \Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor)$$

that agrees on the Zariski open set $\pi^{-1}(\widehat{X}^\circ)$ with the standard pull-back of Kähler differentials.

Proposition 4.7 will be shown below. The phrase "that agrees [. . .] with the standard pull-back of Kähler differentials" might require an explanation.

Explanation 4.8 (Agreeing with the standard pull-back of Kähler differentials). In the setup of Proposition 4.7, consider the non-trivial open set

$$\widehat{X}^\circ = \widehat{X}_{\mathrm{reg}} \cap \gamma^{-1}(X_{\mathrm{reg}} \setminus \operatorname{supp} D)$$

and recall from [KR24a, Ex. 4.6] that on $\widehat{X}^\circ$, the sheaf of adapted reflexive differentials take the simple form

$$\Omega^{[1]}_{(X,D,\gamma)}|_{X^\circ} = \gamma^*\Omega^1_{X_{\mathrm{reg}}}$$

In particular, $\Omega^{[1]}_{(X,D,\gamma)}|_{X^\circ}$ is a subsheaf of the sheaf of Kähler differentials $\Omega^1_{\widehat{X}^\circ}$. A sheaf morphism $d_{\mathcal{C}}\pi$ agrees on $\pi^{-1}(\widehat{X}^\circ)$ with the standard pull-back of Kähler differentials if there exists a commutative diagram of sheaves on $\pi^{-1}(\widehat{X}^\circ)$,

$$\begin{tikzcd}
\pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}\Big|_{\pi^{-1}(\widehat{X}^\circ)} \arrow[r, hook, "d_{\mathcal{C}}\pi"] \arrow[d, hook] & \Omega^1_{\widetilde{X}}(\log \widetilde{D})\Big|_{\pi^{-1}(\widehat{X}^\circ)} \arrow[d, equal] \\
\Big(\pi|_{\pi^{-1}(\widehat{X}^\circ)}\Big)^*\,\Omega^1_{\widehat{X}^\circ} \arrow[r, hook, "d(\pi|_{\pi^{-1}(\widehat{X}^\circ)})", "\text{standard pull-back}"'] & \Omega^1_{\pi^{-1}(\widehat{X}^\circ)}.
\end{tikzcd}$$

Proof of Proposition 4.7. Assume that a sequence of the form (4.1.1) is given. If adapted reflexive p-forms on (X, D) extend, the pull-back of Inclusion (4.3.1) gives a morphism

$$\pi^*\Omega^{[p]}_{(X,D,\gamma)} \subseteq \pi^*\pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) \to \Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor),$$

[2]Locally uniformizable pairs are $\mathcal{C}$-pairs with particularly simple singularities, akin to quotients of pairs with reduced, normal crossing boundary divisor. We refer the reader to [KR24a, Sect. 2.5.1] for the definition and a detailed discussion.

which factors via the reflexive hull $\pi^{[*]}\Omega^{[p]}_{(X,D,\gamma)}$ because the sheaf on the right is locally free. Conversely, if a sheaf morphism of the form $d_{\mathcal{C}}\pi$ is given, we have inclusions

$$\begin{aligned}
\Omega^{[p]}_{(X,D,\gamma)} &\subseteq \pi_*\pi^*\Omega^{[p]}_{(X,D,\gamma)} \subseteq \pi_*\pi^{[*]}\Omega^{[p]}_{(X,D,\gamma)} \\
&\subseteq \pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) && \text{Inclusion } \pi_* d_{\mathcal{C}}\pi \\
&\subseteq \Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor) && \text{Inclusion (4.2.2)}
\end{aligned}$$

as required in Definition 4.3. □

Corollary 4.9 (Extension of adapted reflexive form on locally uniformizable pairs)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. If (X, D) is locally uniformizable, then adapted reflexive p-forms on (X, D) extend to log resolutions of singularities.*

Proof. Given any sequence of the form (4.1.1), recall from [KR24a, Sect. 5] that there exist injective sheaf morphisms

$$d_{\mathcal{C}}\pi : \pi^{[*]}\Omega^{[p]}_{(X,D,\gamma)} \hookrightarrow \Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor)$$

that agree on $\pi^{-1}(\widehat{X}^\circ)$ with the standard pull-back of Kähler differentials. □

4.3. **Extension Criteria.** This section establishes criteria to guarantee extension of adapted reflexive differentials. To begin, we note that extendability is a local property. The elementary proof is left to the reader.

Proposition 4.10 (Local nature of the extension problem)**.** *Let (X, D) be a $\mathcal{C}$-pair, let $p \in \mathbb{N}$ be a number and let $(U_\alpha)_{\alpha\in A}$ be a covering of X by sets that are open in the analytic topology. Then, the following statements are equivalent.*

(4.10.1) Adapted reflexive p-forms on (X, D) extend to log resolutions of singularities.

(4.10.2) Adapted reflexive p-forms on $(U_\alpha, D|_{U_\alpha})$ extend to log resolutions of singularities, for every $\alpha \in A$. □

As an almost immediate corollary of the stratified extension results of Appendix A, we find that adapted reflexive differentials extend if they extend outside a high-codimension subset.

Proposition 4.11 (Restriction to very big open sets)**.** *Let (X, D) be a $\mathcal{C}$-pair, let $p \in \mathbb{N}$ be a number and let $Z \subset X$ be an analytic subset of $\operatorname{codim}_X Z \geq p + 2$. Then, the following statements are equivalent.*

(4.11.1) Adapted reflexive p-forms on (X, D) extend to log resolutions of singularities.

(4.11.2) Adapted reflexive p-forms on $(X \setminus Z, D|_{X\setminus Z})$ extend to log resolutions of singularities.

Proof. Only the implication (4.11.2)⇒(4.11.1) is interesting. Assuming that adapted reflexive p-forms on $(X \setminus Z, D|_{X\setminus Z})$ extend to log resolutions of singularities, consider a sequence of morphisms as in (4.1.1) of Setting 4.1. We need to check Inclusion (4.3.1) of Definition 4.3. To this end, consider the inclusion $\iota : \gamma^{-1}(X\setminus Z) \hookrightarrow \widehat{X}$. Assumption (4.11.2) will then give inclusions between sheaves

$$\iota^*\Omega^{[p]}_{(X,D,\gamma)} \subseteq \iota^*\pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) \subseteq \iota^*\Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor).$$

Since the push-forward ι_* preserves inclusions, we obtain

$$\iota_*\iota^*\Omega^{[p]}_{(X,D,\gamma)} \subseteq \iota_*\iota^*\pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) \subseteq \iota_*\iota^*\Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor). \tag{4.11.3}$$

But reflexivity shows that

$$\begin{aligned}
\iota_*\iota^*\Omega^{[p]}_{(X,D,\gamma)} &= \Omega^{[p]}_{(X,D,\gamma)} \\
\iota_*\iota^*\Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor) &= \Omega^{[p]}_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor)
\end{aligned}$$

and Corollary A.2 of Appendix A gives

$$\iota_*\iota^*\pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor) = \pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor).$$

This presents (4.11.3) as a reformulation of the desired Inclusion (4.3.1). □

Taken at face value, Definition 4.3 ("Extension to log resolutions of singularities") required us to check every sequence of the form (4.1.1). The following proposition simplifies the task, showing that it suffices to consider sequence where the q-morphism γ is adapted.

Proposition 4.12 (Restriction to adapted q-morphisms)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. Assume that for every sequence of morphisms of the following form,*

$$\widetilde{X} \xrightarrow{\rho,\ \text{log resolution of } (\widehat{X}, \delta^*\lfloor D\rfloor)} \widehat{X} \xrightarrow{\delta,\ \text{adapted } q\text{-morphism}} X,$$

there is an inclusion

$$\Omega^{[p]}_{(X,D,\delta)} \subseteq \rho_*\Omega^p_{\widetilde{X}}(\log \rho^*\delta^*\lfloor D\rfloor). \tag{4.12.1}$$

Then, adapted reflexive p-forms on (X, D) extend to log resolutions of singularities.

Proof. Recall from Proposition 4.10 that the extension property is local on X. We may therefore assume without loss of generality that X is Stein and that the divisor D has only finitely many components. By [KR24a, Lem. 2.36], this implies the existence of an adapted cover, say $\mu : \widehat{W} \twoheadrightarrow X$.

In order to verify the conditions spelled out in Definition 4.3, assume that a sequence of morphism is given as in (4.1.1) of Setting 4.1. Next, construct a diagram of the following form,

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, two heads, "\rho,\ \text{equivariant log resolution}"] \arrow[d, two heads, "\widetilde{q},\ \text{gen. finite}"'] & \widehat{Y} \arrow[r, "\delta,\ \text{adapted } q\text{-morphism}"] \arrow[d, two heads, "q,\ \text{Galois cover}"] & X \arrow[d, equal] \\
\widetilde{X} \arrow[r, two heads, "\pi,\ \text{log resolution}"'] & \widehat{X} \arrow[r, "\gamma,\ q\text{-morphism}"'] & X.
\end{tikzcd}$$

To spell the construction out in detail:

- In order to construct the right square, consider the fibre product $\widehat{W} \times_X \widehat{X}$ and take $\widehat{Y}$ as the Galois closure over $\widehat{X}$. The natural morphism δ will then factor via μ and is then, by [KR24a, Obs. 2.27], itself adapted. The natural morphism q is a cover because μ is. Denote the Galois group by G.
- In order to construct the left square, let ρ be a G-equivariant log resolution of the G-variety $\widetilde{X} \times_{\widehat{X}} \widehat{Y}$.

The desired inclusion (4.3.1) will now follow from standard considerations involving G-invariant push-forward:

$$\begin{aligned}
\Omega^{[p]}_{(X,D,\gamma)} &= \left(q_*\Omega^{[p]}_{(X,D,\delta)}\right)^G && \text{by [KR24a, Lem. 4.20]} \\
&\subseteq \left(q_*\rho_*\Omega^p_{\widetilde{Y}}(\log \rho^*\delta^*\lfloor D\rfloor)\right)^G && \text{Assumption (4.12.1)} \\
&= \left(\pi_*\widetilde{q}_*\Omega^p_{\widetilde{Y}}(\log \rho^*\delta^*\lfloor D\rfloor)\right)^G && \text{Commutativity} \\
&= \pi_*\left(\widetilde{q}_*\Omega^p_{\widetilde{Y}}(\log \rho^*\delta^*\lfloor D\rfloor)\right)^G && G\text{-invariance} \\
&= \pi_*\Omega^p_{\widetilde{X}}(\log \pi^*\gamma^*\lfloor D\rfloor)
\end{aligned}$$

The last equality is a standard fact of the geometry of logarithmic pairs, but also follows from [KR24a, Lem. 4.20], using that $\rho^*\delta^*\lfloor D\rfloor$ and $\widetilde{q}^*\pi^*\gamma^*\lfloor D\rfloor$ have identical support. □

We conclude with a proposition showing extension of adapted reflexive p-forms on pairs admitting particularly nice covers, called “perfectly adapted” in [Nú24, Def. 10].

Proposition 4.13 (Extension on existence of good covering spaces)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $p \in \mathbb{N}$ be any number. Assume that there exists a strongly adapted cover $\gamma : \widehat{X} \twoheadrightarrow X$ where $\widehat{X}$ is smooth and $\gamma^* \lfloor D \rfloor$ has snc support. Then, adapted reflexive p-forms on (X, D) extend to log resolutions of singularities.*

Proof. Using the existence of γ, Núñez has shown in [Nú24, Lem. 21] that the conditions of Proposition 4.12 hold. □

4.4. **Proof of Theorem 1.6.** Proposition 4.11 allows removing algebraic subsets $Z \subset X$ of codimension $\operatorname{codim}_X Z \geq 3$. By [Gra15, Prop. 6.19], we can therefore assume that the following dichotomy holds for every point $x \in X$.

(4.14.1) The point x is contained in $X^\circ := X \setminus \operatorname{supp}\lfloor D \rfloor$.
(4.14.2) There exists an algebraic neighbourhood $U = U(x) \subseteq X$ and strongly adapted cover $\gamma : \widehat{U} \twoheadrightarrow U$ where $\widehat{U}$ is smooth and $\gamma^* \lfloor D \rfloor$ has snc support.

Recalling from Proposition 4.10 that the extension problem is local, we may assume that one of the following holds.

(4.14.3) We have $\lfloor D \rfloor = 0$.
(4.14.4) There exists a strongly adapted cover $\gamma : \widehat{X} \twoheadrightarrow X$ where $\widehat{X}$ is smooth and $\gamma^* \lfloor D \rfloor$ has snc support.

In Case (4.14.3), Núñez has shown in [Nú24, Thm. 1] that the assumptions of Proposition 4.12 hold. In Case (4.14.4), the claim follows from Proposition 4.13. □

5. Bogomolov sheaves and linear systems in reflexive G-sheaves

5.1. **The Kodaira dimension for sheaves of adapted reflexive differentials.** Before presenting the main result of the section in Proposition 5.1 below, we recall the notion of “$\mathcal{C}$-Kodaira-Iitaka” dimension in brief for the reader’s convenience. Full details are found in [KR24a, Sects. 4 and 6.2].

5.1.1. *Tensors on manifolds.* If X is a manifold, classic geometry considers the symmetric algebra of tensors on X. Technically speaking, one considers sheaves $\operatorname{Sym}^n \Omega^p_X$ together with symmetric product maps. In particular, if $\mathscr{L} \subseteq \Omega^p_X$ is saturated of rank one, then $\mathscr{L}$ is invertible and the symmetric product sheaf $\operatorname{Sym}^n \mathscr{L}$ is a saturated subsheaf of $\operatorname{Sym}^n \Omega^p_X$.

5.1.2. *Adapted reflexive tensors on $\mathcal{C}$-pairs.* If (X, D) is a $\mathcal{C}$-pair and $\gamma : \widehat{X} \twoheadrightarrow X$ is a cover, the theory of $\mathcal{C}$-pairs considers the symmetric algebra of “adapted reflexive tensors” on $\widehat{X}$. Technically speaking, one considers sheaves $\operatorname{Sym}^{[n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)}$ together with symmetric product maps. In particular, if $\mathscr{L} \subseteq \Omega^{[p]}_{(X,D,\gamma)}$ is saturated of rank one, then $\mathscr{L}$ is reflexive and $\operatorname{Sym}^{[n]} \mathscr{L}$ is a subsheaf $\operatorname{Sym}^{[n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)}$. In contrast to the manifold setting, this sheaf need not be saturated however, and one defines the $\mathcal{C}$-product sheaf

$$\operatorname{Sym}^{[n]}_{\mathcal{C}} \mathscr{L} \subseteq \operatorname{Sym}^{[n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)}$$

as the saturation. One can then consider the set

$$M := \left\{ m \in \mathbb{N} \,\middle|\, h^0\big(\widehat{X}, \operatorname{Sym}^{[m]}_{\mathcal{C}} \mathscr{L}\big) > 0 \right\}$$

and define the *$\mathcal{C}$-Kodaira-Iitaka dimension* as

$$\kappa_{\mathcal{C}}(\mathscr{F}) := \begin{cases} \max_{m \in M} \left\{ \dim \overline{\operatorname{img} \varphi_{\operatorname{Sym}^{[m]}_{\mathcal{C}} \mathscr{L}}} \right\} & \text{if } M \neq \emptyset \\ -\infty & \text{if } M = \emptyset, \end{cases}$$

where $\overline{\bullet}$ denotes Zariski closure in $\mathbb{P}^\bullet$.

5.2. **Comparing C-Kodaira dimensions for sheaves on different covers.** Everything said in Section 5.1.2 depends on the choice of the cover γ. Given that any two covers are dominated by a common third, the main result of the present section explains what happens under a change of covers.

Proposition 5.1 (C-Kodaira dimension of sheaves of adapted differentials)**.** *Let (X, D) be a C-pair, where X is compact. Assume we are given a sequence of covers,*

$$\begin{tikzcd} \widehat{X}_2 \arrow[r, two heads, "\alpha\text{, Galois cover}"'] \arrow[rr, bend left, two heads, "\gamma\text{, cover}"] & \widehat{X}_1 \arrow[r, two heads, "\beta\text{, cover}"'] & X, \end{tikzcd}$$

where α is Galois with group G. If p, d are any two numbers, then the following statements are equivalent.

(5.1.1) There exists a reflexive sheaf $\mathscr{F}_1 \subseteq \Omega^{[p]}_{(X,D,\beta)}$ of rank one with $\kappa_C(\mathscr{F}_1) \geq d$.

(5.1.2) There exists a reflexive G-subsheaf $\mathscr{F}_2 \subseteq \Omega^{[p]}_{(X,D,\gamma)}$ of rank one with $\kappa_C(\mathscr{F}_2) \geq d$.

Proposition 5.1 will be shown in Section 5.4 on page 17. As an immediate corollary, we obtain a slight generalization of Graf's version of the Bogomolov-Sommese vanishing theorem, [Gra15, Thm. 1.2]. Following the literature, we extend the notion of a "Bogomolov sheaf" to this context.

Corollary 5.2 (Bogomolov-Sommese vanishing for G-sheaves of adapted differentials)**.** *Let (X, D) be a log-canonical C-pair where X is compact Kähler. If $\gamma : \widehat{X} \twoheadrightarrow X$ is any cover that is Galois with group G, if p is any number and $\mathscr{F}_1 \subseteq \Omega^{[p]}_{(X,D,\gamma)}$ any G-subsheaf of rank one, then $\kappa_C(\mathscr{F}_1) \leq p$.*

Proof. Apply Proposition 5.1 in case where $\beta = \mathrm{Id}_X$ and recall from [KR24a, Thm. 6.14] that if $\mathscr{F}_2 \subseteq \Omega^{[p]}_{(X,D,\mathrm{Id}_X)}$ is coherent of rank one, then $\kappa_C(\mathscr{F}_2) \leq p$. □

Definition 5.3 (Bogomolov G-sheaf, cf. [KR24a, Def. 6.15])**.** *In the setting of Corollary 5.2, call $\mathscr{F}_1$ a Bogomolov G-sheaf if the equality $\kappa_C(\mathscr{F}_1) = p$ holds.*

The existence of Bogomolov G-sheaves ties in with the notion of "special" C-pairs. We refer the reader to [KR24a, Def. 6.16] for the definition used here and for references to Campana's original work.

Corollary 5.4 (Special pairs have no Bogomolov G-sheaves)**.** *Let (X, D) be a special C-pair. If $\gamma : \widehat{X} \twoheadrightarrow X$ is a Galois cover and if p is any number, then there are no Bogomolov G-sheaves in $\Omega^{[p]}_{(X,D,\gamma)}$.* □

Note that the definition of "special" in [KR24a, Def. 6.16] implies that X is compact Kähler and that (X, D) is log-canonical.

5.3. **Linear systems in reflexive G-sheaves.** To prepare for the proof of Proposition 5.1, the following Proposition 5.6 considers a Galois cover $q : X \twoheadrightarrow Y$, a rank-one G-sheaf $\mathscr{L}$ on X and compares the rational map $\varphi_{\mathscr{L}} : X \dashrightarrow \mathbb{P}^\bullet$ to the rational maps induced by the G-invariant push-forward of the reflexive symmetric products,

$$\varphi_{\left(q_* \operatorname{Sym}^{[n]} \mathscr{L}\right)^G} : Y \dashrightarrow \mathbb{P}^\bullet, \quad \text{for } n \in \mathbb{N}. \tag{5.5.1}$$

To make sense of (5.5.1), recall that the sheaves $\left(q_* \operatorname{Sym}^{[n]} \mathscr{L}\right)^G$ have rank one by construction and are reflexive by [GKKP11, Lem. A.4]. For consistency with the literature we quote, Proposition 5.6 speaks about the reflexive symmetric product of $\mathscr{L}$. Since $\mathscr{L}$ has rank one, we have $\operatorname{Sym}^{[n]} \mathscr{L} = \mathscr{L}^{[\otimes n]}$ and could equally well speak about the reflexive tensor product.

Proposition 5.6. *Let X be a compact, normal analytic variety and let G be a finite group that acts holomorphically on X, with quotient $q : X \twoheadrightarrow X/G$. Let $\mathscr{L}$ be a torsion free G-sheaf of rank one. If $h^0\big(X,\, \mathscr{L}\big) > 0$, then there exists a number $n \in \mathbb{N}^+$ such that*

$$\dim \operatorname{img} \varphi_{\mathscr{L}} \le \dim \operatorname{img} \varphi_{\left(q_* \operatorname{Sym}^{[n]} \mathscr{L}\right)^G}.$$

Proof. The proof is tedious but elementary, and might well be known to experts working in invariant theory. We include full details as we are not aware of a suitable reference.

Step 1: Setup. The group G acts linearly on $L = H^0\big(X,\, \mathscr{L}\big)$ and on its projectivization $\mathbb{P}(L)$, in a way that makes the map $\varphi_{\mathscr{L}}$ equivariant. Consider the quotient $q_{\mathbb{P}} : \mathbb{P}(L) \twoheadrightarrow \mathbb{P}(L)/G$, choose a very ample line bundle $\mathscr{H} \in \operatorname{Pic}\big(\mathbb{P}(L)/G\big)$ and take $n \in \mathbb{N}$ as the unique number satisfying $\mathscr{O}_{\mathbb{P}(L)}(n) = q_{\mathbb{P}}^* \mathscr{H}$. We obtain an identification

$$H^0\big(\mathbb{P}(L)/G,\, \mathscr{H}\big) = H^0\big(\mathbb{P}(L),\, \mathscr{O}_{\mathbb{P}(L)}(n)\big)^G \subseteq H^0\big(\mathbb{P}(L),\, \mathscr{O}_{\mathbb{P}(L)}(n)\big) \tag{5.6.1}$$

and hence a commutative diagram of composable meromorphic maps as follows:

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi_{\mathscr{L}}"] \arrow[d, two heads, "{q,\ \text{finite}}"'] & \mathbb{P}(L) \arrow[r, hook, "{\varphi_{\mathscr{O}_{\mathbb{P}(L)}(n)},\ \text{finite}}"] \arrow[d, two heads, "{q_{\mathbb{P}},\ \text{finite}}"] & \mathbb{P}\Big(H^0\big(\mathbb{P}(L),\, \mathscr{O}_{\mathbb{P}(L)}(n)\big)\Big) \arrow[d, dashed, two heads, "{\mu,\ \text{linear projection}}"] \\
X/G \arrow[r, dashed, "(\varphi_{\mathscr{L}})^G"'] & \mathbb{P}(L)/G \arrow[r, hook, "{\varphi_{\mathscr{H}},\ \text{finite}}"'] & \mathbb{P}\Big(H^0\big(\mathbb{P}(L)/G,\, \mathscr{H}\big)\Big).
\end{tikzcd} \tag{5.6.2}$$

Here,

- the meromorphic map $(\varphi_{\mathscr{L}})^G$ is the induced map between the quotients, given that $\varphi_{\mathscr{L}}$ is G-equivariant,
- the meromorphic map μ is the linear projection induced by (5.6.1) above.

Step 2: Identifications. The linear spaces of Diagram (5.6.2) come with natural identifications and inclusions, which allow expressing some compositions as maps induced by incomplete linear systems. As a variant of (5.6.1), we are interested in the identification

$$H^0\big(\mathbb{P}(L),\, \mathscr{O}_{\mathbb{P}(L)}(n)\big) = \operatorname{Sym}^n L \subseteq H^0\big(X,\, \operatorname{Sym}^{[n]} \mathscr{L}\big), \tag{5.6.3}$$

and its G-invariant version,

$$\begin{aligned}
H^0\big(\mathbb{P}(L)/G,\, \mathscr{H}\big) &= H^0\big(\mathbb{P}(L),\, \mathscr{O}_{\mathbb{P}(L)}(n)\big)^G && \text{by (5.6.1)} \\
&= (\operatorname{Sym}^n L)^G && \text{by (5.6.3)} \\
&\subseteq H^0\big(X,\, \operatorname{Sym}^{[n]} \mathscr{L}\big)^G \\
&= H^0\Big(X/G,\, q_*\big(\operatorname{Sym}^{[n]} \mathscr{L}\big)^G\Big).
\end{aligned} \tag{5.6.4}$$

Observe that (5.6.4) identifies the composed map $\varphi_{\mathscr{H}} \circ (\varphi_{\mathscr{L}})^G$ of Diagram (5.6.2) with the map

$$\varphi_{\left(\operatorname{Sym}^n L\right)^G,\, q_*\left(\operatorname{Sym}^{[n]} \mathscr{L}\right)^G} : X/G \dashrightarrow \mathbb{P}\left(H^0\Big(X/G,\, q_*\big(\operatorname{Sym}^{[n]} \mathscr{L}\big)^G\Big)\right).$$

Step 3: End of proof. In summary, we find

$$\begin{aligned}
\dim \operatorname{img} \varphi_{\mathscr{L}} &= \dim \operatorname{img} \varphi_{\mathscr{H}} \circ q_{\mathbb{P}} \circ \varphi_{\mathscr{L}} && \text{Finiteness} \\
&= \dim \operatorname{img} \varphi_{\mathscr{H}} \circ (\varphi_{\mathscr{H}})^G \circ q && \text{Diagram (5.6.2)} \\
&= \dim \operatorname{img} \varphi_{\mathscr{H}} \circ (\varphi_{\mathscr{H}})^G && \text{Finiteness} \\
&= \dim \operatorname{img} \varphi_{\left(\operatorname{Sym}^n L\right)^G,\, q_*\left(\operatorname{Sym}^{[n]} \mathscr{L}\right)^G} && \text{Step 2} \\
&\le \dim \operatorname{img} \varphi_{q_*\left(\operatorname{Sym}^{[n]} \mathscr{L}\right)^G} && \text{Linear subsystem}
\end{aligned}$$

□

5.4. **Proof of Proposition 5.1.** We consider the two implications separately.

Implication (5.1.1) ⇒ (5.1.2). Given $\mathscr{F}_1 \subseteq \Omega^{[p]}_{(X,D,\beta)}$, set $\mathscr{F}_2 := \alpha^{[*]}\mathscr{F}_1$ and observe that $\mathscr{F}_2$ is indeed a G-subsheaf of the G-sheaf $\Omega^{[p]}_{(X,D,\gamma)}$. Next, recall from [KR24a, Obs. 4.14] that the standard pull-back of Kähler differentials extends to inclusions

$$\alpha^{[*]} \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)} \hookrightarrow \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)}, \quad \text{for every } n \in \mathbb{N}.$$

In particular, we find inclusions

$$\alpha^{[*]} \operatorname{Sym}^{[n]}_C \mathscr{F}_1 \hookrightarrow \operatorname{Sym}^{[n]}_C \alpha^{[*]}\mathscr{F}_1 = \operatorname{Sym}^{[n]}_C \mathscr{F}_2, \quad \text{for every } n \in \mathbb{N}.$$

It follows that $\kappa_C(\mathscr{F}_2) \geq \kappa_C(\mathscr{F}_1) \geq d$, as required. □

Implication (5.1.2) ⇒ (5.1.1). Given a G-subsheaf $\mathscr{F}_2 \subseteq \Omega^{[p]}_{(X,D,\gamma)}$, consider the invariant push-forward sheaves

$$\mathscr{E}_n := \big(\alpha_* \operatorname{Sym}^{[n]}_C \mathscr{F}_2\big)^G, \text{ for every } n \in \mathbb{N}.$$

These sheaves come with inclusions

$$\mathscr{E}_n \subseteq \big(\alpha_* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)}\big)^G = \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)} \qquad \text{for every } n \in \mathbb{N} \tag{5.7.1}$$

$$\mathscr{E}_n \subseteq \operatorname{Sym}^{[n]}_C \mathscr{E}_1 \qquad \text{for every } n \in \mathbb{N}. \tag{5.7.2}$$

The equality in (5.7.1) is shown in [KR24a, Lem. 4.20]. Inclusion (5.7.2) follows because $\operatorname{Sym}^{[n]}_C \mathscr{E}_1$ is saturated inside $\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)}$ by definition, and because the left and right side of the inclusion agree over the dense, Zariski open set

$$\Big(\alpha\big(\widehat{X}_{2,\mathrm{reg}}\big) \cap \widehat{X}_{1,\mathrm{reg}} \cap \beta^{-1}\big(X_{\mathrm{reg}}\big)\Big) \setminus \beta^{-1}(\operatorname{supp} D).$$

With these preparations at hand and taking $\mathscr{F}_1 := \mathscr{E}_1$, we find numbers $n, m \in \mathbb{N}$ such that the following holds.

$$\begin{aligned}
d &\leq \kappa_C(\mathscr{F}_2) && \text{Assumption}\\
&= \dim \operatorname{img} \varphi_{\operatorname{Sym}^{[m]}_C \mathscr{F}_2} && \text{Definition of } \kappa_C\\
&\leq \dim \operatorname{img} \varphi_{\alpha_*\left(\operatorname{Sym}^{[n]} \operatorname{Sym}^{[m]}_C \mathscr{F}_2\right)^G} && \text{Proposition 5.6}\\
&\leq \dim \operatorname{img} \varphi_{\alpha_*\left(\operatorname{Sym}^{[m\cdot n]}_C \mathscr{F}_2\right)^G} && \text{[KR24a, Obs. 4.11]}\\
&\leq \dim \operatorname{img} \varphi_{\operatorname{Sym}^{[m\cdot n]}_C \mathscr{F}_1} && (5.7.2)\\
&\leq \kappa_C(\mathscr{F}_1) && \text{Definition of } \kappa_C \qquad \square
\end{aligned}$$

6. Invariant Bogomolov sheaves defined by strict wedge subspaces

Let X be a compact Kähler manifold and let D be a reduced snc divisor on X. Assume that a finite group G acts on (X, D). Inspired by constructions introduced in [Ran81, Cat91] to generalize the classic Castelnuovo-De Franchis theorem, we show how an abundance of logarithmic one-forms on X can be used to construct G-invariant Bogomolov sheaves, even in cases where no G-invariant differential exists.

6.1. **Strict wedge subspaces.** Strict wedge subspaces, as defined in [Cat91], are the key concept of the present section.

Definition 6.1 (Strict wedge subspaces, [Cat91, Sect. 2]). *Let X be a compact complex manifold and let $D \in \operatorname{Div}(X)$ be a reduced snc divisor. Let $k \in \mathbb{N}$ be a number and $V \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ a linear subspace. We call V a* strict k-wedge subspace of *$H^0\big(X,\, \Omega^1_X(\log D)\big)$ if the following conditions hold.*

- *(6.1.1) The dimension of V is greater than k. In other words, $\dim V > k$.*
- *(6.1.2) The natural map $\wedge^k V \to H^0\big(X,\, \Omega^k_X(\log D)\big)$ is injective.*
- *(6.1.3) The natural map $\wedge^{k+1} V \to H^0\big(X,\, \Omega^{k+1}_X(\log D)\big)$ is identically zero.*

Remark 6.2 (Strict wedge subspaces for special values of k). Assume the setting of Definition 6.1.

- (6.2.1) If $\dim V = k+1$, then every element of $\wedge^k V$ is a pure wedge, and Condition (6.1.2) can be reformulated by saying that no k-tuple of linearly independent 1-forms in V wedges to zero.
- (6.2.2) If $\dim X \le k$, then Condition (6.1.3) holds automatically.

Notation 6.3 (Sheaves of differentials induced by strict wedge spaces). Assume the setting of Definition 6.1. If V is a strict k-wedge subspace of $H^0\big(X,\, \Omega^1_X(\log D)\big)$, write

$$\mathscr{V} := \operatorname{img}\Big(\mathscr{O}_X \otimes V \to \Omega^1_X(\log D)\Big) \subseteq \Omega^1_X(\log D)$$

for the sheaf of 1-forms that can locally be written as linear combinations of forms in V.

Remark 6.4 (Sheaves of differentials induced by strict wedge spaces). Observe that the sheaf $\mathscr{V}$ of Notation 6.3 is (generically) generated by sections in V. Condition (6.1.2) guarantees that its rank equals k.

The following proposition guarantees that strict wedge subspaces exist as soon as there are sufficiently many one-forms.

Proposition 6.5 (Existence of strict wedge subspaces). *Let X be a compact complex manifold, let $D \in \operatorname{Div}(X)$ be a reduced snc divisor, and let $V \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ be a linear subspace of dimension $\dim V > \dim X$. Then, there exists a number $k \in \mathbb{N}^+$ and a strict k-wedge subspace $V' \subseteq V$ of dimension $\dim V' = k+1$.*

Proof. Consider the following set of natural numbers,

$$\begin{aligned} K := \Big\{ \ell \in \mathbb{N} \;:\; & \exists \text{ linearly independent elements } \sigma_1, \dots, \sigma_{\ell+1} \in V \\ & \text{with vanishing } (\ell+1)\text{-form, } \sigma_1 \wedge \cdots \wedge \sigma_{\ell+1} = 0 \in H^0\big(X,\, \Omega^{\ell+1}_X(\log D)\big) \Big\} \subset \mathbb{N}. \end{aligned}$$

Observe that the assumption $\dim V > \dim X$ guarantees that K is not empty. Let k be the minimal element of K, let $\sigma_1, \dots, \sigma_{k+1} \in V$ be linearly independent elements with vanishing $k+1$-form, $\sigma_1 \wedge \cdots \wedge \sigma_{k+1} = 0$, and let V' be the linear subspace spanned by the $\sigma_\bullet$,

$$V' := \langle \sigma_1, \dots, \sigma_k \rangle \subseteq V.$$

The definition of K together with Item (6.2.1) of Remark 6.2 guarantees that V' is indeed a strict k-wedge subspace, as desired. □

6.2. **Linear systems defined by strict wedge subspaces.** If V a strict wedge subspace, the next proposition describes the foliation induced by the meromorphic map η_V introduced in Section 3, effectively bounding its dimension from below. This result is key to all that follows.

Proposition 6.6 (Linear systems defined by strict wedge subspaces)**.** *Let X be a compact Kähler manifold, let $D \in \operatorname{Div}(X)$ be a reduced snc divisor, and let $V \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ be a strict k-wedge subspace. Write $V \subseteq H^0\big(X,\, \mathscr{V}\big)$, where $\mathscr{V} \subseteq \Omega^1_X(\log D)$ is the sheaf introduced in Notation 6.3, and consider the meromorphic map*

$$\eta_V : X \dashrightarrow \mathbb{P}(\operatorname{img} \lambda_V)$$

of Construction 3.2. Then, differentials in V annihilate the foliation $\ker(\eta_V) \subseteq \mathscr{T}_X$ defined by η_V.

Reminder 6.7 (Notation used in Proposition 6.6). We refer the reader to Definitions 2.22 and 2.24 for the precise meaning of the conclusion that "differentials in V annihilate the foliation defined by η_V."

We prove Proposition 6.6 in Section 6.3 on the next page. Before starting the proof, we draw a number of corollaries that will be instrumental when we establish Theorem 1.3 in Section 7 below.

Corollary 6.8 (Linear systems defined by strict wedge subspaces)**.** *In the setup of Proposition 6.6, the image of the rational map η_V has dimension $\dim \operatorname{img} \eta_V = k$.*

Proof. If

$$\mathscr{A} := \ker\Big(\Omega^1_X(\log D) \to \mathscr{H}om_{\mathscr{O}_X}\big(\ker \eta_V,\, \mathscr{O}_X(D)\big)\Big) \subseteq \Omega^1_X(\log D)$$

is the annihilator of the foliation defined by η_V and if $\mathscr{V} \subseteq \Omega^1_X(\log D)$ is the sheaf of differentials introduced in Notation 6.3, then Proposition 6.6 asserts that $\mathscr{V} \subseteq \mathscr{A}$. In particular, we find that

$$k = \operatorname{rank} \mathscr{V} \le \operatorname{rank} \mathscr{A} = \dim \operatorname{img} \eta_V.$$

The converse is shown in Remark 3.4. □

Corollary 6.9 (Linear systems defined by strict wedge subspaces)**.** *Assume the setup of Proposition 6.6. Then, there exists a dense open subset $X^\circ \subseteq X$ where the rational map η_V is well-defined and $\mathscr{V} = \operatorname{img} d\eta_V$.* □

Corollary 6.10 (Sums of sheaves of differentials defined by strict wedge subspaces)**.** *Let X be a compact Kähler manifold, let $D \in \operatorname{Div}(X)$ be a reduced snc divisor, and let $V_1, \dots, V_a \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ be strict wedge subspaces, with associated sheaves $\mathscr{V}_1, \dots, \mathscr{V}_a \subseteq \Omega^1_X(\log D)$ of differentials. Write $\mathscr{V} := \sum_i \mathscr{V}_i \subseteq \Omega^1_X(\log D)$ and consider the determinant $\det \mathscr{V} \subseteq \Omega^{\operatorname{rank}\mathscr{V}}_X(\log D)$. Then,*

$$\operatorname{rank} \mathscr{V} = \dim \operatorname{img} \varphi_{\det \mathscr{V}}.$$

In particular, $\det \mathscr{V}$ is a Bogomolov sheaf in the sense of Remark 3.4.

Proof. Set $V := \sum_i V_i \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ and consider the rational map

$$X \xdashrightarrow{\eta := \eta_{V_1} \times \cdots \times \eta_{V_a}} \mathbb{P}\Big(\operatorname{img} \lambda_{V_1}\Big) \times \cdots \times \mathbb{P}\Big(\operatorname{img} \lambda_{V_a}\Big).$$

Combining results obtained so far, there exists an open subset $X^\circ \subseteq X$ where all maps encountered so far are well-defined and the following chain of (in)equalities holds.

$$\begin{aligned}
\operatorname{rank} \mathscr{V} = \operatorname{rank} \textstyle\sum_i \mathscr{V}_i|_{X^\circ} &= \operatorname{rank} \textstyle\sum_i \operatorname{img}(d\eta_{V_i}|_{X^\circ}) && \text{Corollary 6.9} \\
&= \operatorname{rank} \operatorname{img}(d\eta|_{X^\circ}) && \text{Lemma 2.7} \\
&= \dim \operatorname{img} \eta && \\
&\le \dim \operatorname{img} \eta_V && \text{Corollary 3.8} \\
&\le \dim \operatorname{img} \varphi_{\det \mathscr{V}} && \text{Remark 3.4}
\end{aligned}$$

The converse inequality, $\dim \operatorname{img} \varphi_{\det \mathscr{V}} \le \operatorname{rank} \mathscr{V}$, holds by Remark 3.4. □

Corollary 6.11 (Invariant Bogomolov sheaves defined by strict wedge subspaces)**.** *Let X be a compact Kähler manifold, equipped with a holomorphic action of a (possibly infinite) group G. Let $D \in \operatorname{Div}(X)$ be a reduced snc divisor that is G-stable. Assume that X admits a strict wedge subspace $V \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ with associated sheaf $\mathscr{V}$ of differentials, and write $\mathscr{W} := \sum_{g \in G} g^* \mathscr{V}$. Then,*

$$\operatorname{rank} \mathscr{W} = \dim \operatorname{img} \varphi_{\det \mathscr{W}}.$$

In particular, $\det\big(\mathscr{W}\big) \subseteq \Omega^{\operatorname{rank} \mathscr{W}}_X(\log D)$ *is a G-invariant Bogomolov sheaf.*

Proof. The Noetherian condition guarantees that $\sum_{g \in G} g^* \mathscr{V}$ is in fact a finite sum. □

6.3. **Proof of Proposition 6.6.** For the reader's convenience, we subdivide the proof into steps.

Step 0: Simplification and Notation. The definition of "strict k-wedge subspace" guarantees that every linear subspace $V' \subseteq V$ is again a strict k-wedge subspace, as long as $\dim V' > k$. The factorization found in Proposition 3.5,

$$\begin{tikzcd} X \arrow[r, dashed, "\eta_V"'] \arrow[rr, dashed, bend left, "\eta_{V'}"] & \mathbb{P}\Big(\operatorname{img} \lambda_V\Big) \arrow[r, dashed, two heads, "\exists \eta_{V,V'}"'] & \mathbb{P}\Big(\operatorname{img} \lambda_{V'}\Big), \end{tikzcd}$$

equips us with an inclusion of the foliations defined by η_V and $\eta_{V'}$,

$$\ker(\eta_V) \subseteq \ker(\eta_{V'}),$$

which allows us to assume the following.

Assumption w.l.o.g. 6.12. The dimension of V equals $k + 1$.

Resolving the indeterminacies of η_V by a suitable blow-up, $\pi : \widetilde{X} \to X$ and replacing $V \subseteq H^0\big(X,\, \Omega^1_X(\log D)\big)$ by its pull-back $(d\pi)(V) \subseteq H^0\big(\widetilde{X},\, \Omega^1_{\widetilde{X}}(\log \pi^* D)\big)$, we may assume the following.

Assumption w.l.o.g. 6.13. The meromorphic mapping η_V is in fact holomorphic.

Denote the image variety by $Y := \operatorname{img} \eta_V \subseteq \mathbb{P}(\operatorname{img} \lambda_V)$ and let $Y^\circ \subset Y$ be the maximal open subset over which the morphism η_V is a proper submersion.

Step 1: Coordinates. We will work with explicit coordinates and choose an ordered basis $\sigma_1, \dots, \sigma_{k+1} \in V$. A natural ordered basis of $\wedge^k V$ is then given as

$$(\sigma_1 \wedge \cdots \wedge \underbrace{\sigma_j}_{\text{delete}} \wedge \cdots \wedge \sigma_{k+1})_{1 \le j \le k+1} \in \wedge^k V.$$

Use this basis to identify $\mathbb{P}(\operatorname{img} \lambda_V) \cong \mathbb{P}(\wedge^k V) \cong \mathbb{P}^k$ and use homogeneous coordinates $[x_1 : \cdots : x_{k+1}]$ to denote its points. This choice of coordinates allows writing η_V in terms of concrete functions. To make this statement precise, observe that the forms $\sigma_1, \dots, \sigma_{k+1}$ span the rank-k sheaf $\mathscr{V}$ generically. Consequently, there exist meromorphic functions $\alpha_1, \dots, \alpha_k \in \mathscr{M}(X)$ such that

$$\sigma_{k+1} = \sum_{j=1}^{k} \alpha_j \cdot \sigma_j. \tag{6.14.1}$$

This implies in particular that

$$\sigma_1 \wedge \cdots \wedge \underbrace{\sigma_j}_{\text{delete}} \wedge \cdots \wedge \sigma_{k+1} = (-1)^{k-j} \cdot \alpha_j \cdot \sigma_1 \wedge \cdots \wedge \sigma_k, \quad \text{for every index } j \le k.$$

On the open set $U \subseteq X$ where the $\alpha_1, \dots, \alpha_k$ are regular, the morphism η_V is thus described as

$$\eta_V|_U : U \to \mathbb{P}^k, \quad x \mapsto \big[(-1)^{k-1} \cdot \alpha_1(x) : \cdots : (-1)^{k-k} \cdot \alpha_k(x) : 1\big].$$

Step 2: Forms on Fibres. Given any point $s \in Y^\circ \setminus \{x_{k+1} = 0\}$, write $s = [y_1 : \cdots : y_k : 1]$ and consider the holomorphic differential form

$$\tau_s := \sigma_{k+1} - \sum_{i=1}^{k} (-1)^{k-i} \cdot y_i \cdot \sigma_i.$$

Linear independence of $\sigma_1, \ldots, \sigma_{k+1}$ guarantees that τ_s does not vanish identically on X. Equation (6.14.1) however guarantees that τ_s vanishes identically along the fibre $X_s := \eta_V^{-1}(s)$. Proposition 2.21 then guarantees that τ_s annihilates the foliation defined by η_V over the open set $\eta_V^{-1}(Y^\circ)$ and hence everywhere.

Step 3: Non-degeneracy of the image. The construction of η_V guarantees that the image variety Y is *not* linearly degenerate inside $\mathbb{P}^k$. In other words, Y is not contained in any linear hyperplane. If $(s_1, \ldots, s_k) \in \big(Y^\circ \setminus \{x_{k+1} = 0\}\big)^{\times k}$ is a general k-uple of points, written as $s_i = [y_{i1} : \cdots : y_{ik} : 1]$, then non-degeneracy implies that

$$(y_{11}, \ldots, y_{1k}, 1), \ldots, (y_{(k)1}, \ldots, y_{(k)k}, 1) \in \mathbb{C}^{k+1}$$

are linearly independent vectors. The holomorphic forms

$$\tau_{s_i} := \sigma_{k+1} - \sum_{j=1}^{k} (-1)^{k-i} \cdot y_{ij} \cdot \sigma_j$$

are likewise linearly independent, hence form a basis of V, and annihilate the foliation defined by η_V. □

7. Bounds on invariants, Proof of Theorem 1.3 and Corollary 1.7

7.1. **Proof of Theorem 1.3.** We prove the contrapositive: assuming that (X, D) is an n-dimensional C-pair with irregularity $q^+(X, D) > n$ satisfying the assumptions of Theorem 1.3, we will show that (X, D) is *not* special.

Step 1: Setup. The assumption that $q^+(X, D) > n$ allows choosing a Galois cover $\gamma : \widehat{X} \to X$ with irregularity $q(X, D, \gamma) > n$. Fix one such cover throughout and denote its Galois group by G. Consider the reduced divisor $\widehat{D} := (\gamma^* \lfloor D \rfloor)_{\mathrm{red}}$ and let $\pi : \widetilde{X} \to \widehat{X}$ be a G-equivariant, strict log resolution of the pair $(\widehat{X}, \widehat{D})$,

$$\widetilde{X} \xrightarrow{\pi,\ \text{resolution}} \widehat{X} \xrightarrow{\gamma,\ \text{cover}} X. \tag{7.1.1}$$

The preimage $\pi^{-1}\big(\operatorname{supp} \widehat{D}\big)$ is then G-invariant, of pure codimension one and has simple normal crossing support. Let $\widetilde{D} \in \operatorname{Div}\big(\widetilde{X}\big)$ be the associated divisor. The pair $(\widetilde{X}, \widetilde{D})$ is snc, and the sheaf $\Omega^1_{\widetilde{X}}(\log \widetilde{D})$ is a G-sheaf.

Claim 7.2. There exists an injective sheaf morphism

$$d_C\pi : \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \hookrightarrow \Omega^1_{\widetilde{X}}(\log \widetilde{D})$$

that agrees on the Zariski open set $\pi^{-1}(\widehat{X}^\circ)$ with the standard pull-back of Kähler differentials, in the sense discussed in Explanation 4.8.

Proof of Claim 7.2. We consider the alternative assumptions of Theorem 1.3 separately. In case (1.3.1), where (X, D) is locally uniformizable, this is [KR24a, Fact 5.9]. In case (1.3.2), where X is projective and (X, D) is dlt, this is Theorem 1.6, in the formulation given by Proposition 4.7. □ (Claim 7.2)

By minor abuse of notation, we suppress $d_C\pi$ from the notation and view $\pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}$ as a G-subsheaf of the G-sheaf $\Omega^1_{\widetilde{X}}(\log \widetilde{D})$.

Step 2: An invariant Bogomolov sheaf on $\widetilde{X}$. We know by assumption that

$$h^0\big(\widetilde{X},\, \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}\big) = h^0\big(\widehat{X},\, \Omega^{[1]}_{(X,D,\gamma)}\big) = q(X,D,\gamma) > n.$$

Use Proposition 6.5 to find a number p and a strict p-wedge subspace

$$V \subseteq H^0\big(\widetilde{X},\, \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}\big) \subseteq H^0\big(\widetilde{X},\, \Omega^1_{\widetilde{X}}(\log \widetilde{D})\big)$$

with the associated sheaf $\mathscr{V} \subseteq \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \subseteq \Omega^1_{\widetilde{X}}(\log \widetilde{D})$ of differentials. Denoting the sum of the pull-back sheaves by

$$\mathscr{W} := \sum_{g\in G} g^*\mathscr{V} \subseteq \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \subseteq \Omega^1_{\widetilde{X}}(\log \widetilde{D}),$$

Corollary 6.11 shows that

$$\det(\mathscr{W}) \subseteq \wedge^{[\operatorname{rank}\mathscr{W}]}\pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \subseteq \Omega^{\operatorname{rank}\mathscr{W}}_{\widetilde{X}}(\log \widetilde{D}) \tag{7.3.1}$$

satisfies $\operatorname{rank}\mathscr{W} \le \dim \operatorname{img}\varphi_{\det \mathscr{W}}$ and is hence a G-invariant Bogomolov sheaf.

Step 3: An invariant Bogomolov sheaf on $\widehat{X}$. We conclude the proof of Theorem 1.3 by exhibiting the push-forward $\pi_* \det \mathscr{W}$ as a G-invariant Bogomolov sheaf on $\widehat{X}$. The next claim allows viewing the push-forwards as a sheaf of adapted reflexive differentials.

Claim 7.4. There exists a natural inclusion of G-sheaves,

$$\pi_* \wedge^{[\operatorname{rank}\mathscr{W}]} \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \hookrightarrow \Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}.$$

Proof of Claim 7.4. The sheaf on the left is the push-forward of a torsion free sheaf, and hence itself torsion free. It agrees with the sheaf on the right over the big open set $(\widehat{X},\widehat{D})_{\mathrm{reg}} \subseteq \widehat{X}$ where the strict resolution map π is isomorphic. In other words: denoting the inclusion by $\iota : (\widehat{X},\widehat{D})_{\mathrm{reg}} \hookrightarrow \widehat{X}$, we find an isomorphism

$$\iota^*\pi_* \wedge^{[\operatorname{rank}\mathscr{W}]} \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \xrightarrow{\text{isomorphic}} \iota^*\Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}.$$

Pushing forward, this gives a sequence of sheaf morphisms,

$$\pi_* \wedge^{[\operatorname{rank}\mathscr{W}]} \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \hookrightarrow \iota_*\iota^*\pi_* \wedge^{[\operatorname{rank}\mathscr{W}]} \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \twoheadrightarrow \iota_*\iota^*\Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)},$$

where injectivity of the first arrow comes from the fact that $\pi_* \wedge^{[\operatorname{rank}\mathscr{W}]} \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}$ is torsion free. To conclude, it suffices to note that $\Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}$ is reflexive, and hence equal to the last term in the sequence, $\iota_*\iota^*\Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}$. □ (Claim 7.4)

Inclusion (7.3.1) and Claim 7.4 equip us with an inclusion of G-sheaves,

$$\pi_* \det \mathscr{W} \subseteq \Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}.$$

The natural isomorphism $H^0\big(\widetilde{X},\, \det\mathscr{W}\big) = H^0\big(\widehat{X},\, \pi_*\det\mathscr{W}\big)$ will then show that

$$\operatorname{rank}\mathscr{W} \le \dim\operatorname{img}\varphi_{\det\mathscr{W}} = \dim\operatorname{img}\varphi_{\pi_*\det\mathscr{W}}.$$

The sheaf $\pi_*\det\mathscr{W}$ is thus a G-invariant Bogomolov sheaf, and the claim follows from Corollary 5.4. This finishes the proof of Theorem 1.3. □

7.2. **Proof of Corollary 1.7.** Let (X,D) be a projective $\mathcal{C}$-pair that is dlt. Assuming the existence of a cover $\gamma : \widehat{X} \twoheadrightarrow X$ and a rank-one sheaf $\mathscr{L} \subseteq \Omega^{[1]}_{(X,D,\gamma)}$ of positive $\mathcal{C}$-Kodaira-Iitaka dimension, $\kappa_{\mathcal{C}}(\mathscr{L}) > 0$, we need to show that (X,D) is *not* special. The proof is largely parallel to the proof of Theorem 1.3 given in the previous Section 7.1.

Step 0: Simplifying assumptions. Given a sequence of covers,

$$\begin{tikzcd} \check{X} \arrow[r, two heads, "\beta\text{, Galois cover}"'] \arrow[rr, bend left, two heads, "\alpha\text{, cover}"] & \widehat{X} \arrow[r, two heads, "\gamma\text{, cover}"'] & X, \end{tikzcd} \tag{7.5.1}$$

Proposition 5.1 equips us with rank-one, reflexive subsheaf $\check{\mathscr{L}} \subseteq \Omega^{[1]}_{(X,D,\gamma\circ\beta)}$ on $\check{X}$ with positive $\mathcal{C}$-Kodaira-Iitaka dimension, $\kappa_{\mathcal{C}}(\check{\mathscr{L}}) \geq \kappa_{\mathcal{C}}(\mathscr{L}) > 0$. We will use this observation to replace the cover γ by higher covers with additional properties.

Step 0.1: The cover is adapted. The projectivity assumption and [KR24a, Lem. 2.36] imply that X admits an adapted cover. An elementary fibre product construction will then allow us to produce a Sequence (7.5.1) of covers where α is itself adapted.

Assumption w.l.o.g. 7.6. The cover γ is adapted and Galois, with Group G.

Step 0.2: The sheaf $\mathscr{L}$ admits sections. The assumption on the positivity of the $\mathcal{C}$-Kodaira-Iitaka dimension, $\kappa_{\mathcal{C}}(\mathscr{L}) > 0$, equips us with number $m > 0$ and two linearly independent sections

$$\sigma_0, \sigma_1 \in H^0\big(\widehat{X}, \operatorname{Sym}^{[m]}_{\mathcal{C}} \mathscr{L}\big). \tag{7.7.1}$$

The right side of (7.7.1) simplifies because of Assumption 7.6. To be precise, recall from [KR24a, Obs. 4.12] that

$$\operatorname{Sym}^{[m]}_{\mathcal{C}} \mathscr{L} = \mathscr{L}^{[\otimes m]} \quad \text{so that} \quad \sigma_0, \sigma_1 \in H^0\big(\widehat{X}, \mathscr{L}^{[\otimes m]}\big).$$

Standard covering constructions ("taking the m-th root out of a section") produce a Sequence (7.5.1) of covers and sections

$$\tau_0, \tau_1 \in H^0\big(\check{X}, \beta^{[*]}\mathscr{L}\big)$$

whose m-th tensor powers agree over a dense open set with the pullbacks of σ_0 and σ_1. Recalling from [KR24a, Obs. 4.8] that the sheaf $\beta^{[*]}\mathscr{L}$ injects into $\Omega^{[1]}_{(X,D,\gamma\circ\beta)}$, we may replace γ by α and make the following assumption.

Assumption w.l.o.g. 7.8. The sheaf $\mathscr{L}$ admits two linearly independent sections.

Step 1: Setup. Consider the reduced divisor $\widehat{D} := (\gamma^*\lfloor D\rfloor)_{\mathrm{red}}$ and let $\pi : \widetilde{X} \to \widehat{X}$ be a G-equivariant, strict log resolution of the pair $(\widehat{X}, \widehat{D})$, as in (7.1.1) above. Taking $\widetilde{D} \in \operatorname{Div}(\widetilde{X})$ as the divisor supported on $\pi^{-1}(\operatorname{supp}\widehat{D})$, we find the following.

Claim 7.9 (Analogue of Claim 7.2). There exists an injective sheaf morphism

$$d_{\mathcal{C}}\pi : \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \hookrightarrow \Omega^1_{\widetilde{X}}(\log \widetilde{D})$$

that agrees on the Zariski open set $\pi^{-1}(\widehat{X}^\circ)$ with the standard pull-back of Kähler differentials, in the sense discussed in Explanation 4.8. □

Suppress $d_{\mathcal{C}}\pi$ from the notation and view $\pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}$ as a G-subsheaf of the G-sheaf $\Omega^1_{\widetilde{X}}(\log \widetilde{D})$.

Step 2: An invariant Bogomolov sheaf on $\widetilde{X}$. Use Assumption 7.8 to define a strict 1-wedge subspace

$$V := \langle \tau_0, \tau_1\rangle \subseteq H^0\big(\widehat{X}, \Omega^{[1]}_{(X,D,\gamma)}\big) \subseteq H^0\big(\widetilde{X}, \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)}\big) \overset{\text{Claim 7.9}}{\subseteq} H^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D})\big)$$

with the associated sheaf $\mathscr{V} \subseteq \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \subseteq \Omega^1_{\widetilde{X}}(\log \widetilde{D})$ of differentials. Denoting the sum of the pull-back sheaves by

$$\mathscr{W} := \sum_{g\in G} g^*\mathscr{V} \subseteq \pi^{[*]}\Omega^{[1]}_{(X,D,\gamma)} \subseteq \Omega^1_{\widetilde{X}}(\log \widetilde{D}),$$

Corollary 6.11 shows that $\det(\mathscr{W}) \subseteq \Omega^{\operatorname{rank}\mathscr{W}}_{\widetilde{X}}(\log \widetilde{D})$ satisfies $\operatorname{rank}\mathscr{W} \leq \dim \operatorname{img} \varphi_{\det \mathscr{W}}$ and is hence a G-invariant Bogomolov sheaf.

Step 3: An invariant Bogomolov sheaf on $\widehat{X}$. In analogy with the proof of Theorem 1.3, observe that there exists a natural inclusion of G-sheaves, $\pi_* \det \mathscr{W} \subseteq \Omega^{[\operatorname{rank}\mathscr{W}]}_{(X,D,\gamma)}$, that exhibits $\pi_* \det \mathscr{W}$ is as a G-Bogomolov sheaf. This finishes the proof of Corollary 1.7. □

Appendix

Appendix A. Extension of low degree differentials

Let X be a normal analytic variety and let $\pi : \widetilde{X} \to X$ be a resolution of singularities. If the singular set of X is small, $\operatorname{codim}_X X_{\operatorname{sing}} \geq p + 2$ for one $p \in \mathbb{N}$, then Flenner has shown in [Fle88] that p-forms extend from the smooth locus of X to p-forms on $\widetilde{X}$: The natural restriction map

$$H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) \hookrightarrow H^0\big(\pi^{-1}(X_{\operatorname{reg}}),\, \Omega^p_{\widetilde{X}}\big) = H^0\big(X_{\operatorname{reg}},\, \Omega^p_X\big)$$

is isomorphic. Writing $\iota : X_{\operatorname{reg}} \hookrightarrow X$ for the inclusion map, Flenner's result can equivalently be stated by saying the natural inclusion

$$\pi_* \Omega^p_{\widetilde{X}} \to \iota_* \iota^* \pi_* \Omega^p_{\widetilde{X}}$$

is isomorphic. The proof builds on earlier work [vSS85] of Steenbrink and van Straten. It relies on relative duality for cohomology with supports and Hodge-theoretic methods.

A.1. **Main Result.** The present section extends Flenner's result to forms with logarithmic poles and replaces X_{sing} with arbitrary subvarieties $Z \subseteq X$: If Z is small, $\operatorname{codim}_X Z \geq p + 2$ for one $p \in \mathbb{N}$, then p-forms extend from $\widetilde{X} \setminus \pi^{-1}(Z)$ to p-forms on $\widetilde{X}$.

Theorem A.1 (Extension of low degree differentials)**.** *Let (X, D) be a logarithmic pair, let $\pi : \widetilde{X} \to X$ be a log resolution of singularities, and let $\widetilde{D} \in \operatorname{Div}(\widetilde{X})$ be the reduced divisor supported on $\pi^{-1}D$. Let $p \in \mathbb{N}$ be any number. If $Z \subset X$ is a Zariski closed subset of $\operatorname{codim}_X Z \geq p + 2$, then the natural restriction map*

$$H^0\left(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right) \hookrightarrow H^0\left(\widetilde{X} \setminus \pi^{-1}(Z),\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right) \tag{A.1.1}$$

is isomorphic.

Corollary A.2 (Extension of low degree differentials)**.** *In the setting of Theorem A.1, write $\iota : X \setminus Z \to X$ for the inclusion map. Then, natural morphism*

$$\pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{D}) \to \iota_* \iota^* \pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{D})$$

is isomorphic. □

Theorem A.1 will be shown in Section A.4 below.

A.2. **Idea of application.** Theorem A.1 will typically be used in scenarios where the singular locus X_{sing} comes with a stratification and Z is a (potentially strict) subvariety of X_{sing}. One can then consider restrictions,

$$\begin{aligned} H^0\left(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right) &\overset{\alpha}{\hookrightarrow} H^0\left(\widetilde{X} \setminus \pi^{-1}(Z),\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right) \\ &\overset{\beta}{\hookrightarrow} H^0\left(\pi^{-1}(X_{\operatorname{reg}}),\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right) = H^0\left(X_{\operatorname{reg}},\, \Omega^p_X(\log D)\right) \end{aligned}$$

and ask if a given form $\sigma^\circ \in H^0\left(X_{\operatorname{reg}},\, \Omega^p_X(\log D)\right)$ is induced by a form on $\widetilde{X}$. If one knows that σ° lies in the image of β (for instance, because the singularities of X are sufficiently mild outside the stratum Z), then Theorem A.1 might apply to show that σ° already lies in the image of $\beta \circ \alpha$.

These ideas are put to work in Section 4.3, where we prove Theorem 1.6. There, $Z \subset X_{\mathrm{sing}}$ is a stratum of high codimension, such that X has no worse than quotient singularities outside Z.

A.3. **Relation to the literature.** To the best of our knowledge, Theorem A.1 has not appeared in the literature before. The arguments used in the proof are however not new, and likely known among experts. Our proof combines ideas of Graf, who replaces duality for cohomology with supports by numerical computation involving Kodaira dimension and Bogomolov-Sommese vanishing, with work of Núñez, who follows [KS21] by expressing extension properties in terms of the dimension of the support of certain local cohomology sheaves.

Despite appearance to the contrary, the results presented here are unrelated to the general extension results of [GKKP11, KS21] that are much more difficult to prove.

A.4. **Proof of Theorem A.1.** For completeness' sake, we give a full and mostly self-contained proof, combining results and ideas from [Gra21, Nú24]. The reader will note that Steps 3–5 are copied with only minor modifications from [Gra21, p. 599f].

Step 1: Setup. The restriction map (A.1.1) is clearly injective. To show surjectivity, let

$$\sigma^\circ \in H^0\left(\widetilde{X} \setminus \pi^{-1}(Z),\ \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right)$$

be any form. We need to find a differential form $\sigma \in H^0\left(\widetilde{X},\ \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right)$ whose restriction to $\widetilde{X} \setminus \pi^{-1}(Z)$ agrees with σ°.

Step 2: Simplification. The case where $D \in \operatorname{Div}(X)$ is the zero divisor has been worked out by Núñez in [Nú24, Lem. 24]. Applying Núñez' result to $X \setminus \operatorname{supp} D$, we already obtain a form on $\widetilde{X} \setminus \pi^{-1}(Z \cap D)$ whose restriction to $\widetilde{X} \setminus \pi^{-1}(Z)$ agrees with σ°. This allows making the following assumption.

Assumption w.l.o.g. A.3. The set Z is contained in the support of D.

Since any two log resolutions of singularities are dominated by a common third, it is clear that validity of Theorem A.1 does not depend on the choice of the log resolution π. Replacing π by a different resolution if need be, we may assume the following.

Assumption w.l.o.g. A.4. The preimage $\pi^{-1}(Z)$ is of pure codimension one in $\widetilde{X}$ and has simple normal crossing support.

Write $E \in \operatorname{Div}(\widetilde{X})$ for the reduced divisor supported on $\pi^{-1}(Z)$.

Step 3: Extension as a rational form. Grauert's Direct Image Theorem, [GR84, Chapt. 10.4], implies that the sheaf $\pi_* \Omega^1_{\widetilde{X}}(\log \widetilde{D})$ is coherent. The pull-back σ° therefore extends as a meromorphic log-form τ on $\widetilde{X}$. Quantifying the poles, let $G \in \operatorname{Div}(\widetilde{X})$ be the minimal effective divisor such that there exists a section

$$\tau \in H^0\left(\widetilde{X},\ \Omega^p_{\widetilde{X}}(\log \widetilde{D})(G)\right) = \operatorname{Hom}\left(\mathscr{O}_{\widetilde{X}}(-G),\ \Omega^p_{\widetilde{X}}(\log \widetilde{D})\right)$$

whose restriction to $\widetilde{X} \setminus \pi^{-1}(Z)$ agrees with σ°. Aiming to show that $G = 0$, we argue by contradiction.

Assumption A.5. The divisor G is not zero.

Minimality of G implies that G is π-exceptional. Since π is a log resolution, G will then have simple normal crossing support. Assumption A.5 and [Gra21, Prop. 4] imply the existence of an irreducible component $P \subseteq \operatorname{supp} G$ such that the invertible sheaf $\mathscr{O}_{\widetilde{X}}(-G)$ is big when restricted a general fibre of $\pi|_P$. Minimality of G guarantees that the restricted map of sheaves on P,

$$\tau|_P : \mathscr{O}_{\widetilde{X}}(-G)\big|_P \to \Omega^p_{\widetilde{X}}(\log \widetilde{D})\Big|_P,$$

does not vanish.

Step 4: Residue sequence. Considering the divisor $P^c := (E-P)|_P \in \operatorname{Div}(P)$, the residue sequence for p-forms along P reads

$$\begin{tikzcd} & & \mathscr{O}_{\widetilde{X}}(-G)\big|_P \arrow[d, "\tau|_P"] & & \\ 0 \arrow[r] & \Omega^p_P(\log P^c) \arrow[r] & \Omega^p_{\widetilde{X}}(\log E)\Big|_P \arrow[r, "\operatorname{res}_P"'] & \Omega^{p-1}_P(\log P^c) \arrow[r] & 0. \end{tikzcd}$$

We obtain an injection

$$\mu : \mathscr{O}_{\widetilde{X}}(-G)|_P \hookrightarrow \Omega^r_P(\log P^c),$$

for one $r \in \{p-1, p\}$. In both cases, $r \le p$.

Step 5: Restriction to the general fibre. For brevity of notation, write $B := \pi(P)$ and denote the restricted morphism $\pi|_P$ by $\rho : P \to B$. Next, let $F \subset P$ be a general fibre of ρ and consider the restricted divisor $F^c := P^c|_F \in \operatorname{Div}(F)$. Since (P, P^c) is an snc pair, so is (F, F^c). The inequality $\operatorname{codim}_X Z \ge p+2$ implies that

$$\begin{aligned} \dim F &= \dim P - \dim B \\ &\ge (\dim X - 1) - \dim Z \\ &\ge (\dim X - 1) - (\dim X - (p+2)) = p+1. \end{aligned} \tag{A.6.1}$$

Since F is general and $\rho(F)$ is a smooth point of B, the restriction of the standard sequence of relative log differentials, [EV92, Sec. 4.1], reads

$$0 \to \rho^*\Omega^1_B\big|_F \to \Omega^1_P(\log P^c)\big|_F \to \Omega^1_{P/B}(\log P^c)\Big|_F \to 0.$$

By [Har77, Ch. II, Ex. 5.16], it induces a filtration of $\Omega^r_P(\log P^c)|_F$ with quotients

$$\rho^*\Omega^i_B \otimes \Omega^{r-i}_{P/B}(\log P^c), \quad i = 0, \dots, r.$$

Restricting the morphism μ to the general fibre F, we obtain an injection

$$\mu|_F : \mathscr{O}_{\widetilde{X}}(-G)|_F \hookrightarrow \rho^*\Omega^i_B \otimes \Omega^{r-i}_{P/B}(\log P^c) = \Omega^{r-i}_F(\log F^c)^{\oplus \operatorname{rank} \Omega^i_B},$$

for one suitable number i. Projecting onto a suitable summand, we even find an inclusion

$$\mathscr{O}_{\widetilde{X}}(-G)|_F \hookrightarrow \Omega^{r-i}_F(\log F^c).$$

This leads to a contradiction.

- On the one hand, the classic Bogomolov-Sommese vanishing theorem, [EV92, Cor. 6.9], implies that $\kappa\left(\mathscr{O}_{\widetilde{X}}(-G)|_F\right) \le r - i \le r \le p$.
- On the other hand, the choice of G guarantees that $\mathscr{O}_{\widetilde{X}}(-G)$ is big when restricted a general fibre of $\pi|_P$, so that $\kappa\left(\mathscr{O}_{\widetilde{X}}(-G)|_F\right) = \dim F$. However, we have seen in (A.6.1) that $\dim F \ge p+1$.

Assumption A.5 is therefore absurd. We conclude that $G = 0$, as desired. ◻

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