ENTIRE CURVES IN $\mathcal{C}$-PAIRS WITH LARGE IRREGULARITY

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Abstract. This paper extends the fundamental theorem of Bloch-Ochiai to the context of $\mathcal{C}$-pairs: If (X, D) is a $\mathcal{C}$-pair with large irregularity, then no entire $\mathcal{C}$-curve in X is ever dense. In its most general form, the paper's main theorem applies to normal Kähler pairs with arbitrary singularities. However, it also strengthens known results for compact Kähler manifolds without boundary, as it applies to some settings that the classic Bloch-Ochiai theorem does not address.

The proof builds on work of Kawamata, Ueno, and Noguchi, recasting parabolic Nevanlinna theory as a "Nevanlinna theory for $\mathcal{C}$-pairs". We hope the approach might be of independent interest.

Contents

1. Hyperbolicity properties of pairs with large irregularity

1.1. **Degeneracy of entire curves.** The Albanese variety is a fundamental tool in the study of entire curves (or rational points) in projective varieties. Its usefulness is illustrated by the Bloch-Ochiai Theorem.

Theorem 1.1 (Bloch-Ochiai Theorem, [Kaw80, Thm. 2])**.** *Let X be a projective manifold. If the irregularity of X is larger than the dimension, $q(X) > \dim X$, then every entire curve $\mathbb{C} \to X$ is algebraically degenerate.* □

Reminder 1.2. An entire curve is a holomorphic morphism $\mathbb{C} \to X$. An entire curve is algebraically degenerate if the image of $\mathbb{C}$ is not Zariski dense in X.

We recall the main ideas of the proof: start with the Albanese morphism $a : X \to A$ and let $I \subseteq a(X)$ be the largest Abelian subvariety of A whose action stabilizes $\operatorname{img}(a)$ and consider the quotient $B := A/I$. The image of X in B is then of general type, which reduces the problem to a study of entire curves in general type subvarieties of Abelian varieties. With these preparations, the Bloch-Ochiai Theorem 1.1 is then an easy consequence of the following result, which follows for instance from [NW14, Thms. 4.8.2 and 2.5.4][1].

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[1] See [NW14, p. 155] for further explanation.

Theorem 1.3 (Entire curves in varieties of general type)**.** *Let W be a projective manifold of general type. If the Albanese morphism* $\operatorname{alb}(W) : W \to \operatorname{Alb}(W)$ *is generically injective, then every entire curve* $\mathbb{C} \to W$ *is algebraically degenerate.* □

Theorems 1.1 and 1.3 have both been generalized to the setting of logarithmic pairs (X, D), where X is a compact Kähler manifold and D is a reduced divisor (not necessarily with simple normal crossing support). We refer the reader to [NW14, Sect. 4.8 and Thm. 4.8.17] for precise statements and for a brief history of the problem.

1.2. **Degeneracy of C-entire curves.** This paper generalizes Theorem 1.1 to the setting of C-pairs (X, D), where X is a normal Kähler space and D is a Weil $\mathbb{Q}$-divisor of the form

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i, \quad \text{all } m_i \in \mathbb{N}^{\geq 2} \cup \{\infty\}.$$

Originally introduced under the name "geometric orbifold" by Campana, C-pairs interpolate between compact spaces and spaces with logarithmic boundary. By now, C-pairs and the derived notions such as "adapted differentials" and "C-cotangent sheaves" are standard tools in complex, algebraic and arithmetic geometry, with applications ranging from the geometry of moduli spaces to the study of rational points over function fields, [CP19, KPS22]. We recall the relevant notions in brief and refer the reader to [KR24b, KR24a] for references and a detailed introduction.

Definition 1.4 (Entire C-curve, algebraic degeneracy)**.** *Let (X, D) be a C-pair. An* entire C-curve *is a holomorphic morphism of C-pairs,* $(\mathbb{C}, 0) \to (X, D)$*. An entire C-curve is* algebraically degenerate *if the image of $\mathbb{C}$ is not Zariski dense in X.*

Reminder 1.5 (Morphisms of C-pairs). Morphisms of C-pairs are formally introduced and discussed at length in [KR24b, Sect. 7ff]. In the simplest case, where X is smooth and D is a smooth prime divisor of the form

$$D = \frac{m_1 - 1}{m_1} \cdot D_1, \quad \text{for } m_1 \in \mathbb{N}^{\geq 2},$$

C-pairs impose tangency conditions reminiscent of (but different from) the morphism $\gamma : \mathbb{C} \to X$ is a C-morphism between the C-pairs $(\mathbb{C}, 0)$ and tangency conditions imposed by root stacks, [Cad07]: a holomorphic (X, D) if

$$\gamma^* D \geq m_i \cdot \operatorname{supp}(\gamma^* D).$$

In other words, every intersection point of curve γ and the divisor D must have multiplicity m_i at least (and not necessarily divisible by m_i as in the case of root stacks).

Reminder 1.6 (Augmented Albanese irregularity). Let (X, D) be a C-pair where X is a compact Kähler space. Generalizing the *augmented irregularity* of a compact manifold, [KR24a, Sect. 5.1] introduces the "augmented Albanese irregularity" $q^+_{\operatorname{Alb}}(X, D)$, with values in $\mathbb{N} \cup \{\infty\}$.

Theorem 1.7 (C-version of the Bloch-Ochiai theorem, see Proposition 5.2)**.** *Let (X, D) be a C-pair where X is a compact Kähler space. If $q^+_{\operatorname{Alb}}(X, D) > \dim X$, then every C-entire curve $(\mathbb{C}, 0) \to (X, D)$ is algebraically degenerate.*

Remark 1.8 (Assumptions in Theorem 1.7). Theorem 1.7 does *not* assume that X is smooth, but the definition of C-pair does require that X is normal, [KR24b, Def. 2.13]. Theorem 1.7 explicitly covers the case where $q^+_{\operatorname{Alb}}(X, D) = \infty$.

Remark 1.9 (Comparison with Theorem 1.1). The irregularity of a compact Kähler manifold X is never larger than the augmented Albanese irregularity of the trivial C-pair $(X, 0)$,

$$q(X) \leq q^+_{\operatorname{Alb}}(X, 0).$$

Examples show that the inequality might be strict. Even in case where X is smooth and $D = 0$, Theorem 1.7 is therefore a priori stronger than Theorem 1.1 (and its logarithmic version [NW14, Thm. 4.8.17]) and could prove hyperbolicity in settings where Theorem 1.1 does not apply.

Remark 1.10 (Theorem 1.7 and the existence of an Albanese). Recall from [KR24a, Sect. 9] that a $\mathcal{C}$-pair has an Albanese with good universal properties if and only if its augmented Albanese irregularity is finite. In this sense, Theorem 1.7 can be seen as saying that either

- an Albanese of $\mathcal{C}$-pair (X, D) exists and has rather small dimension, or
- (X, D) has strong hyperbolicity properties.

At its core, our proof of Theorem 1.7 re-interprets parabolic Nevanlinna theory as a "Nevanlinna theory for $\mathcal{C}$-pairs". We hope that the reader might find this of independent interest.

1.3. **Perspective: Specialness and $\mathcal{C}$-entire curves.** To put Theorem 1.7 into perspective, we recall a famous conjecture of Campana that relates specialness to the existence of dense entire $\mathcal{C}$-curves.

Conjecture 1.11 (Specialness and $\mathcal{C}$-entire curves, [Cam11, Conj. 13.17]). Let (X, D) be a snc $\mathcal{C}$-pair where X is projective or compact Kähler. Then, the pair (X, D) is special if and only if it admits a Zariski-dense entire $\mathcal{C}$-curve.

Remark 1.12. Even if one is only interested in the statement for varieties (that is, the case where $D = 0$), the use of the word "special" implicitly turns Conjecture 1.11 into a statement about $\mathcal{C}$-pairs. Any result towards Conjecture 1.11 will necessarily need to take $\mathcal{C}$-pairs into account.

If (X, D) is a special Kähler $\mathcal{C}$-pair, we have seen in [KR24a, Rem. 7.4] that the augmented irregularity is bounded by the dimension, $q^+_{\mathrm{Alb}}(X, D) \leq \dim X$. In particular, Conjecture 1.11 predicts that $\mathcal{C}$-pairs with $q^+_{\mathrm{Alb}}(X, D) > \dim X$ have no Zariski dense entire curves. This is exactly the content of Theorem 1.7. In cases where X is smooth and D is empty or where D is reduced, this is exactly the logarithmic analogue of the Bloch-Ochiai Theorem 1.1. We refer the reader to [NW14, Thm. 4.8.17] for details and for a discussion.

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The proof of Theorem 1.7 follows ideas of Kawamata, builds on work of Ueno and uses Nevanlinna theory, as developed by Noguchi and others.

1.5. **Global conventions.** This paper works in the category of complex analytic spaces and follows the notation of the standard reference texts [GR84, Dem12, NW14]. An *analytic variety* is a reduced, irreducible complex space. For clarity, we refer to holomorphic maps between analytic varieties as *morphisms* and reserve the word *map* for meromorphic mappings.

We use the language of $\mathcal{C}$-pairs, as surveyed in [KR24b], and freely refer to definitions and results from [KR24b] throughout the present text. The same holds for the paper [KR24a], which introduces the core notion of a $\mathcal{C}$-semitoric pair and constructs the Albanese of a $\mathcal{C}$-pair. The reader might wish to keep hardcopies of both papers within reach.

2. Cyclic group actions on semitoric varieties and differentials

We will later need several elementary statements about actions of finite, cyclic groups on semitoric varieties. While certainly known to experts, we were not able to find a suitable reference and include a full proof below. We refer the reader to [NW14, Def. 5.3.3] for the definition of "semitoric varieties", and to [KR24a, Sect. 3] for a detailed discussion.

Setting and Notation 2.1. Let $A^\circ \subset A$ be a positive-dimensional semitoric variety, and let $G \subset \operatorname{Aut}(A, \Delta_A)$ be a non-trivial, finite, cyclic group. Then, G acts on the space of logarithmic differentials and decomposes this space into a direct sum of eigenspaces. More precisely, there exists an identification $G = \mathbb{Z}/(\operatorname{ord} G)$ and a unique decomposition

$$H^0\big(A,\ \Omega^1_A(\log \Delta_A)\big) = \bigoplus_{0 \le \lambda < \operatorname{ord} G} E_{G,\lambda}, \tag{2.1.1}$$

where G acts on every summand $E_{G,\lambda}$ by homotheties of the form

$$\mathbb{Z}/(\operatorname{ord} G) \times E_{G,\lambda} \to E_{G,\lambda}, \quad \big([\ell], \tau\big) \mapsto \exp\left(\ell \cdot \lambda \cdot \frac{2\pi}{\operatorname{ord} G} \cdot \sqrt{-1}\right) \cdot \tau. \tag{2.1.2}$$

Fix the identification throughout. Recalling from [KR24a, Prop. 3.15] that the sheaf $\Omega^1_A(\log \Delta_A)$ is free, the decomposition (2.1.1) induces a decomposition of sheaves,

$$\Omega^1_A(\log \Delta_A) = \bigoplus \mathscr{E}_{G,\lambda} \quad \text{and} \quad \mathscr{T}_A(-\log \Delta_A) = \bigoplus \mathscr{E}^*_{G,\lambda}. \tag{2.1.3}$$

Remark 2.2. The summands $\mathscr{E}^*_{G,\bullet}$ of Setting 2.1 are free. They can equivalently be described as

$$\mathscr{E}^*_{G,\lambda} = \bigcap_{\mu \ne \lambda} \ \bigcap_{\tau \in E_{G,\mu}} \ker \tau.$$

Remark 2.3. The summands $\mathscr{E}^*_{G,\bullet}$ of Setting 2.1 are invariant under the action of A°. Since A° is commutative as a Lie group, its Lie bracket vanishes and the restriction of every summand to A° is a foliation.

If the cyclic group G of Setting 2.1 acts on A by translations with elements of A°, then the induced action on the space of differentials is trivial and $H^0\big(A,\ \Omega^1_A(\log \Delta_A)\big) = E_{G,0}$. Since this is hardly interesting, we concentrate on the case where G has a fixed point and more relevant statements can be made. The following result is certainly not the best possible, but suffices for our purposes.

Lemma 2.4. *Assume Setting 2.1. If the G-action on A° has a fixed point, then the leaves of $\mathscr{E}^*_{G,0}|_{A^\circ}$ are contained in the translates of proper, quasi-algebraic sub-semitori of A°.*

Proof. Fix an element $0_{A^\circ} \in A^\circ$ in order to equip A° with the structure of a Lie group. Using that the foliation $\mathscr{E}^*_{G,0}|_{A^\circ}$ is translation-invariant, it suffices to show that the leaf through 0_{A° is contained in a proper, quasi-algebraic sub-semitorus of A°. To this end, choose one G-fixed point $a \in A^\circ$, choose a generator $h \in G$ and write

$$\eta := L_a^{-1} \circ h \circ L_a \in \operatorname{Aut}(A, \Delta_A),$$

where $L_a \in \operatorname{Aut}(A, \Delta_A)$ is the translation that sends 0_{A° to a. Observe that the element $\eta \in \operatorname{Aut}(A, \Delta_A)$ fixes 0_{A°. The elements h and η have the same order, and the associated cyclic groups $G = \langle h \rangle$ and $\langle \eta \rangle$ are thus canonically isomorphic. Since translations act trivially on differentials, the actions of G and $\langle \eta \rangle$ on $H^0\big(A,\ \Omega^1_A(\log \Delta_A)\big)$ agree under this identification. To prove Lemma 2.4, we can therefore replace G by $\langle \eta \rangle$ and assume without loss of generality that $G \subset \operatorname{Aut}(A, \Delta_A)$ fixes the point $0_{A^\circ} \in A$ and therefore acts linearly on the tangent space $T_{A^\circ}|_{\{0_{A^\circ}\}}$. We know what the action is: The Decomposition (2.1.3) induces a decomposition

$$T_{A^\circ}|_{\{0_{A^\circ}\}} = \bigoplus T_{G,\lambda},$$

and G acts on every summand by homotheties of the form

$$\mathbb{Z}/(\operatorname{ord} G) \times T_{G,\lambda} \to T_{G,\lambda}, \quad \big([\ell], \vec{v}\big) \mapsto \exp\left(-\ell \cdot \lambda \cdot \frac{2\pi}{\operatorname{ord} G} \cdot \sqrt{-1}\right) \cdot \vec{v}.$$

In particular, G acts trivially on the summand $T_{G,0}$.

The exponential morphism $\exp : T_{A^\circ}|_{\{0_{A^\circ}\}} \to A^\circ$ of the Lie group A° is a surjective, locally biholomorphic group morphism that is equivariant for the actions of G on $T_{A^\circ}|_{\{0_{A^\circ}\}}$

and on A°, respectively. The image $\exp(T_{G,0})$ equals the leaf of $\mathscr{E}^*_{G,0}$ through 0. But the equivariant exponential morphism sends G-fixed points to G-fixed points. Recalling from [KR24a, Prop. 3.18] that $h|_{A^\circ} \in \operatorname{Aut}(A^\circ)$ is a group morphism, this means that the leaf of $\mathscr{E}^*_{G,0}$ through 0_{A° is then necessarily contained in

$$\operatorname{Fix}(G) \cap A^\circ = \ker\big(h|_{A^\circ} - \operatorname{Id}_{A^\circ}\big) \subseteq A^\circ.$$

Recall from [KR24a, Facts 3.23 and 3.27] that this is indeed a quasi-algebraic, proper sub-semitorus of A°. □

3. Nevanlinna theory for branched covers of $\mathbb{C}$

To prepare for the proof of Theorem 1.7, this section recalls a number of useful results from Nevanlinna theory. We refer the reader to [Yam15, Sect. 3 and p. 250f] and [NW14, Sect. 2.7] for details and for a well-written introduction to Nevanlinna theory for branched covers of $\mathbb{C}$. To begin, we fix setting and notation for the remainder of the present section.

Setting 3.1 (Holomorphic cover of the complex plane). Let V be a connected Riemann surface and $\rho : V \twoheadrightarrow \mathbb{C}$ be a cover (recall the convention [KR24b, Def. 2.21] that covers are finite). We denote the standard coordinate function on the complex line by $t \in H^0\big(\mathbb{C},\, \mathscr{O}_{\mathbb{C}}\big)$. Given any real number $r \geq 0$, let $\Delta_r \subset \mathbb{C}$ be the disk of radius r and write $V_r := \rho^{-1}(\Delta_r)$ for its preimage.

3.1. The Nevanlinna functions.

Maintain Setting 3.1. Aiming to generalize Bloch-Ochiai's Theorem 1.1, we are interested in a criterion to guarantee that holomorphic morphisms from V to a projective manifold Y are algebraically degenerate. The criterion, Theorem 4.1 on page 9, builds on work of Noguchi and makes heavy use the "main Nevanlinna functions for the branched covering ρ". We recall the definitions of the Nevanlinna functions and briefly state their main properties and refer to [NW14, Sect. 2.7] for a more detailed introduction.

Reminder 3.2 (Counting functions). In Setting 3.1, let $H \in \operatorname{Div}(V)$ be any effective divisor. We can then consider the following functions.

$$N_H : [1,\infty) \to \mathbb{R}^{\geq 0}, \quad r \mapsto \frac{1}{\deg \rho} \int_1^r \Big(\sum_{u \in V_s} \operatorname{ord}_u H \Big) \frac{ds}{s} \qquad \text{Counting}$$

$$N_{1,H} : [1,\infty) \to \mathbb{R}^{\geq 0}, \quad r \mapsto \frac{1}{\deg \rho} \int_1^r \Big(\sum_{u \in V_s} \min\{1, \operatorname{ord}_u H\} \Big) \frac{ds}{s} \qquad \text{Truncated counting}$$

Reminder 3.3 (Proximity and height functions). In Setting 3.1, let $g : V \to Y$ be any non-constant morphism from V to a projective manifold Y, equipped with a Hermitian line bundle $L := \big(\mathscr{L}, |\cdot|\big)$ and a section $\sigma \in H^0\big(Y, \mathscr{L}\big)$ such that $\sigma \circ g$ is not identically zero. Writing $c_1(L)$ for the Chern form of the Hermitian bundle L, we consider the following functions,

$$m(\bullet, g, L, \sigma) : [1,\infty) \to \mathbb{R}, \quad r \mapsto \frac{1}{\deg \rho} \int_{\partial V_r} \log \frac{1}{|\sigma \circ g|} \cdot \rho^*(d^c \log |t|^2) \qquad \text{Proximity}$$

$$T(\bullet, g, L) : [1,\infty) \to \mathbb{R}, \quad r \mapsto \frac{1}{\deg \rho} \int_1^r \Big(\int_{V_s} g^* c_1(L) \Big) \frac{ds}{s} \qquad \text{Height}$$

Remark 3.4 (Integral in the proximity function). The existence of the integral in the definition of $m(\bullet, g, L, \sigma)$ is elementary, cf. [NW14, (2.3.30) and Sect. 2.7]. For the reader's convenience, we remark that our main reference, [NW14], writes $\gamma := d^c \log |t|^2$. Our second main reference, [Nog85], uses the notation $\eta := \rho^*(d^c \log |t|^2)$. We will constantly use the fact that

$$\int_{\partial \Delta_r} d^c \log |t|^2 = 1 \quad \text{and hence} \quad \int_{\partial V_r} \rho^*(d^c \log |t|^2) = \deg \rho. \tag{3.4.1}$$

The following elementary facts are well-known to experts, cf. [Yam15, p. 234 and 250]. We include full proofs for the reader's convenience.

Lemma 3.5 (Boundedness of the proximity function)**.** *The function $m(\bullet, g, L, \sigma)$ of Reminder 3.3 is bounded from below.*

Proof. This follows from Equation (3.4.1), given that the continuous function $Y \to \mathbb{R}^{\geq 0}$, $y \to |\sigma(y)|$ on the compact space Y is bounded from above. □

Lemma 3.6 (Independence on choice of metric)**.** *If the bundle $\mathscr{L}$ of Reminder 3.3 carries two Hermitian metrics, $L_1 := \big(\mathscr{L}, |\cdot|_1\big)$ and $L_2 := \big(\mathscr{L}, |\cdot|_2\big)$, then*

$$m(\bullet, g, L_1, \sigma) = m(\bullet, g, L_2, \sigma) + O(1) \tag{3.6.1}$$

$$T(\bullet, g, L_1) = T(\bullet, g, L_2) + O(1). \tag{3.6.2}$$

Proof. Equation (3.6.1) follows from (3.4.1), given that the two norm functions $|\cdot|_1$ and $|\cdot|_2 \in C^0(\mathscr{L})$ differ only by multiplication with the pull-back of a strictly positive function in $C^0(V)$, which attains its minimum and maximum.

The proof of (3.6.2) is almost identical to [Yam15, proof of Lem. 3.1]. To begin, observe that $c_1(L_1)$ and $c_1(L_2)$ are smooth closed $(1,1)$-forms on Y with identical cohomology class. The difference $c_1(L_1) - c_1(L_2)$ is thus exact, and the $\partial\bar{\partial}$-lemma yields a smooth function φ on V such that $c_1(L_1) - c_1(L_2) = dd^c\varphi$. We find

$$\begin{aligned}
&T(r, g, L_1) - T(r, g, L_2) \\
&\quad = \frac{1}{\deg\rho}\int_1^r \left(\int_{V_s} g^*\big(c_1(L_1) - c_1(L_2)\big)\right)\frac{ds}{s} \\
&\quad = \frac{1}{\deg\rho}\int_1^r \left(\int_{V_s} dd^c(\varphi\circ g)\right)\frac{ds}{s} \\
&\quad = \frac{1}{\deg\rho}\int_1^r \left(\int_{\partial V_s} d^c(\varphi\circ g)\right)\frac{ds}{s} && \text{Stokes} \\
&\quad = \frac{1}{\deg\rho}\int_{V_r\setminus V_1} d^c(\varphi\circ g)\wedge\frac{d|t\circ\rho|}{|t\circ\rho|} && \text{Fubini} \\
&\quad = \frac{1}{2\cdot\deg\rho}\int_{V_r\setminus V_1} d^c(\varphi\circ g)\wedge\rho^*(d\log|t|^2) \\
&\quad = \frac{-1}{2\cdot\deg\rho}\int_{V_r\setminus V_1} d(\varphi\circ g)\wedge\rho^*(d^c\log|t|^2) && d^cu\wedge dv = d^cv\wedge du \\
&\quad = \frac{-1}{2\cdot\deg\rho}\int_{V_r\setminus V_1} d\Big((\varphi\circ g)\cdot\rho^*(d^c\log|t|^2)\Big) && dd^c\log|t|^2 = 0 \\
&\quad = \frac{-1}{2\cdot\deg\rho}\left(\int_{\partial V_r}(\varphi\circ g)\cdot\rho^*(d^c\log|t|^2)\right. \\
&\qquad\qquad \left. - \int_{\partial V_1}(\varphi\circ g)\cdot\rho^*(d^c\log|t|^2)\right) && \text{Stokes.}
\end{aligned}$$

Since φ is bounded as a continuous function on the compact manifold Y, Equation (3.4.1) implies that the integrals in the last line are bounded. □

Lemma 3.7 (Height function for ample divisor)**.** *If the bundle $\mathscr{L}$ of Reminder 3.3 is ample, then the height function tends to infinity. More precisely, there exist $c_1^+ \in \mathbb{R}^+$ and $c_2 \in \mathbb{R}$ such that*

$$T(r, g, L) \geq c_1^+ \cdot \log r + c_2, \quad \textit{for every } r \geq 1.$$

Proof. Ampleness of $\mathscr{L}$ and Lemma 3.6 allows replacing $|\cdot|$ with a metric of positive Chern form. The proof of [Yam15, p. 234] will then apply verbatim:

$$T(r,g,L) = \frac{1}{\deg\rho}\int_1^r\left(\int_{V_s} g^*c_1(L)\right)\frac{ds}{s} \geq \frac{1}{\deg\rho}\int_1^r\left(\int_{V_1} g^*c_1(L)\right)\frac{ds}{s} = \text{const}^+\cdot\log r \quad\square$$

We also recall that the Nevanlinna functions of Reminders 3.2 and 3.3 are related to one another by the following fundamental result.

Theorem 3.8 (First main theorem, cf. [NW14, Thm. 2.7.4])**.** *In the setting of Reminder 3.3, let $D \in \operatorname{Div}(Y)$ be the zero-divisor of the section σ. Then,*

$$T(\bullet,g,L) = N_{g^*D}(\bullet) + m(\bullet,g,L,\sigma) + O(1). \qquad\square$$

3.2. **The Lemma on logarithmic derivatives.** The next section develops a degeneracy criterion, Theorem 4.1, whose proof uses a fundamental fact of Nevanlinna theory for branched covers of $\mathbb{C}$: the "Lemma on logarithmic derivatives". For the reader's convenience, we briefly recall the statement. The following notation will be used to compare differentials on V with the standard differential $\mathrm{d}\,t$ on the complex plane.

Notation 3.9 (Differentials on V and the standard differential on the complex line). In Setting 3.1, observe that every meromorphic differential $\tau \in H^0\big(V,\,\Omega^1_V\otimes\mathscr{K}_V\big)$ can be written as $\xi\cdot(\rho^*dt)$, where $\xi\in H^0\big(V,\mathscr{K}_V\big)$ is meromorphic. Writing $\xi =: \frac{\tau}{\rho^*dt}$ for ease of notation, we can thus define a morphism that takes meromorphic differentials to meromorphic functions,

$$\eta : H^0\big(V,\,\Omega^1_V\otimes\mathscr{K}_V\big)\to H^0\big(V,\mathscr{K}_V\big),\qquad \tau\mapsto\frac{\tau}{\rho^*dt}.$$

The Lemma on logarithmic derivatives views the meromorphic functions ξ as morphisms $\xi : V \to \mathbb{P}^1$ and considers the proximity function with respect to the standard Hermitian structure on the hyperplane bundle of $\mathbb{P}^1$. The following notation will be used.

Notation 3.10 (Hermitian structure on the anti-tautological bundle). Denote the standard Hermitian structure on the hyperplane bundle of $\mathbb{P}^1$ by $H := (\mathscr{O}_{\mathbb{P}^1}(1),|\cdot|)$. Writing z for the standard coordinate on $\mathbb{C}\subset\mathbb{P}^1$, we also consider the standard sections $\sigma_0,\sigma_\infty\in H^0\big(\mathbb{P}^1,\,\mathscr{O}_{\mathbb{P}^1}(1)\big)$, where $\operatorname{div}\sigma_\bullet = \bullet$, where $\sigma_0 = z\cdot\sigma_\infty$ and

$$|\sigma_0(z)|^2 = \frac{|z|^2}{|z|^2+1}\quad\text{and}\quad |\sigma_\infty(z)|^2 = \frac{1}{|z|^2+1}. \tag{3.10.1}$$

Theorem 3.11 (Lemma on logarithmic derivatives, [Nog85, Lem. 1.6])**.** *In the setting of Reminder 3.3, assume that $\mathscr{L}$ is ample. Given a reduced divisor $D\in\operatorname{Div}(Y)$ with $\operatorname{img} g\not\subset \operatorname{supp} D$ and a logarithmic differential $\omega\in H^0\big(Y,\,\Omega^1_Y(\log D)\big)$, consider the meromorphic function $\xi := \eta\big(g^*\omega\big)$, and view it as a morphism $\xi : V\to\mathbb{P}^1$. If $\varepsilon>0$ is any number, there exists an inequality of the following form,*

$$m(\bullet,\xi,H,\sigma_\infty)\leq\varepsilon\cdot T(\bullet,g,L)\qquad\|. \tag{3.11.1}$$

Reminder 3.12 (Notation used in (3.11.1)). As usual in Nevanlinna theory, the symbol $\|$ in (3.11.1) means that the inequality holds outside a subset of $[1,\infty)$ that is a union of (possibly infinitely many) intervals with finite total measure. The subset may well depend on the number ε.

Proof of Theorem 3.11. Theorem 3.11 is a reformulation of [Nog85, Lem. 1.6]. To begin, observe that it follows from Lemma 3.7 that the validity of Inequality (3.11.1) depends only on the classes of the functions $m(\bullet,\xi,H,\sigma_\infty)$ and $T(\bullet,g,L)$, modulo addition of bounded functions. We use this freedom in two ways.

- Using Lemma 3.6, we may replace the Hermitian metric on the ample bundle $\mathscr{L}$ and assume without loss of generality that $c_1(L)$ is a positive form on V. This will later become relevant when we invoke [Nog85, Lem. 1.6], where positivity of $c_1(L)$ is an implicit assumption[2].

- We may replace the proximity function $m(\bullet, \xi, H, \sigma_\infty)$ in (3.11.1) with the simpler variant $m(\bullet, \xi)$ used in Noguchi's paper.

We explain the second bullet item in detail and consider the estimates

$$\begin{aligned}
&(3.13.1) \quad & m(r,\xi,H,\sigma_\infty) &= \frac{1}{\deg \rho}\int_{\partial V_r} \log \frac{1}{|\sigma_\infty \circ \xi|}\cdot \rho^*(d^c \log|t|^2) && \text{definition}\\
&(3.13.2) \quad & &= \frac{1}{\deg \rho}\int_{\partial V_r} \log \sqrt{|\xi|^2+1}\cdot \rho^*(d^c \log|t|^2) && (3.10.1)\\
&(3.13.3) \quad & &= \underbrace{\frac{1}{\deg \rho}\int_{\partial V_r} \log^+ |\xi|\cdot \rho^*(d^c \log|t|^2)}_{=:m(r,\xi),\text{ as defined in [Nog85, p. 298]}} + O(1), && \text{see below}
\end{aligned}$$

where

$$\log^+ : \mathbb{R} \to \mathbb{R}^{\geq 0}, \quad r \mapsto \begin{cases} 0 & \text{if } r < 1 \\ \log r & \text{otherwise.} \end{cases}$$

The estimate (3.13.3) follows from (3.4.1) and from the elementary inequality

$$0 \leq \log\sqrt{r^2+1} - \log^+ r \leq \log\sqrt{2}, \quad \text{for every } r \in \mathbb{R}^{\geq 0}.$$

Wrapping up what we have shown so far: To prove Theorem 3.11, it suffices to show that for every $\varepsilon > 0$, there exists an inequality of the form

$$(3.13.4) \qquad m(\bullet, \xi) \leq \varepsilon \cdot T(\bullet, g, L) \quad \|.$$

choose $\delta \in (0,1)$ such that $\delta \leq \varepsilon \cdot c_1^+$ and recall from [Nog85, Lem. 1.6 and Given one ε, consider the constants c_1^+ and c_2 of Lemma 3.7, proof on p. 302] that there exists a constant $c \in \mathbb{R}$ and an inequality of the form

$$(3.13.5) \qquad m(\bullet, \xi) \leq \delta \cdot \log \bullet + 20 \cdot \log^+ T(\bullet, g, L) + c \quad \|.$$

But given that $T(\bullet, g, L)$ is monotonous and unbounded, the following will hold for all sufficiently large numbers $r \gg 0$,

$$(3.13.6) \qquad 0 \leq 20 \cdot \log^+ T(r,g,L) \leq \frac{\varepsilon}{3}\cdot T(r,g,L),$$

$$(3.13.7) \qquad c - \frac{\delta \cdot c_2}{c_1^+} \leq \frac{\varepsilon}{3}\cdot T(r,g,L).$$

For these numbers sufficiently large numbers r, the right side of (3.13.5) then reads

$$\begin{aligned}
&\delta \cdot \log r + 20\cdot \log^+ T(r,g,L) + c \\
&\quad = \frac{\delta}{c_1^+}(c_1^+ \cdot \log r) + 20 \cdot \log^+ T(r,g,L) + c \\
&\quad \leq \frac{\delta}{c_1^+}\cdot T(r,g,L) + 20\cdot \log^+ T(r,g,L) + c - \frac{\delta\cdot c_2}{c_1^+} && \text{Lem. 3.7}\\
&\quad \leq \frac{\varepsilon}{3}\cdot T(r,g,L) + 20\cdot \log^+ T(r,g,L) + c - \frac{\delta\cdot c_2}{c_1^+} && \text{choice of } \delta \text{ and (3.13.6)}\\
&\quad \leq \varepsilon \cdot T(r,g,L) && \text{(3.13.6) and (3.13.7).}
\end{aligned}$$

This establishes an inequality of the desired form (3.13.4) and finishes the proof of Theorem 3.11. □

[2] The sentence "we assume that Ω is the positive form associated with a Hermitian metric h on X" in [Nog85, p. 299] contains a misprint. The symbol "X" should read "V".

4. A degeneracy criterion for entire curves

Building on work of Noguchi, this section establishes a criterion to guarantee algebraic degeneracy of the morphism g from Reminder 3.3.

Theorem 4.1 (Degeneracy criterion). *In the setting of Reminder 3.3, let $D \in \operatorname{Div}(Y)$ be a reduced divisor with snc support, such that the following holds.*

(4.1.1) The Albanese morphism $\operatorname{alb}(Y,D)^\circ$ of the log pair is generically finite.
(4.1.2) The image of $\operatorname{alb}(Y,D)^\circ$ is a variety of log-general type.
(4.1.3) The image of g does not intersect D.

Suppose that there exists a reduced divisor[3] $D_1 \in \operatorname{Div}(Y)$ with $\operatorname{img} g \not\subset \operatorname{supp} D_1$ and logarithmic differentials $\omega_1, \dots, \omega_l \in H^0\big(Y,\, \Omega^1_Y(\log D_1)\big)$ such that the associated meromorphic functions $\xi_i := \eta\big(g^*\omega_i\big)$ are holomorphic and do not vanish identically. If

$$\operatorname{supp}\big(\operatorname{Ramification}\rho\big) \subseteq \bigcup_{i\in\{1,\dots,l\}} \{v \in V \,:\, \xi_i(v) = 0\}, \tag{4.1.4}$$

then g is algebraically degenerate.

Explanation 4.2. Condition (4.1.2) might require a word of explanation. To formulate the condition precisely, choose one Albanese and consider the map

$$\operatorname{alb}(Y,D)^\circ : Y^\circ \to \operatorname{Alb}(Y,D)^\circ \subset \operatorname{Alb}(Y,D).$$

Consider the toplogical closure $W := \overline{\operatorname{img}\operatorname{alb}(Y,D)^\circ}$. Observe that W is analytic because $\operatorname{alb}(Y,D)^\circ$ is quasi-algebraic, and write $W^\circ := W \cap \operatorname{Alb}(Y,D)^\circ$. We obtain a tuple (W,Δ) where W is a (potentially non-normal) variety and $\Delta = W \setminus W^\circ$ is an analytic subset of pure codimension one. Condition (4.1.2) says that one (equivalently: every) log-resolution of (W,Δ) is of log-general type. Since $\operatorname{Alb}(Y,D)$ is unique up to bimeromorphic equivalence, this condition does not depend on the choice made in the construction.

We prove Theorem 4.1 in Section 4.2 below.

4.1. Noguchi's criterion.

The proof of Theorem 4.1 relies on the following proposition. Essentially due to Noguchi, it replaces Condition (4.1.4) by an inequality between Nevanlinna functions. The interested reader might also want to look at a related criterion of Yamanoi, [Yam10, Prop. 3.3], that is stronger but works only in the compact case.

Proposition 4.3 (Noguchi's criterion). *In the setting of Reminder 3.3, let $D \in \operatorname{Div}(Y)$ be a reduced divisor with snc support, such that the following holds.*

(4.3.1) The Albanese morphism $\operatorname{alb}(Y,D)^\circ$ of the log pair (Y,D) is generically finite.
(4.3.2) The image of $\operatorname{alb}(Y,D)^\circ$ is a variety of log-general type.
(4.3.3) The image of g does not intersect D.

If the line bundle $\mathscr{L} \in \operatorname{Pic}(Y)$ is ample and if the inequality

$$N_{\operatorname{Ramification}\rho}(\bullet) \le \varepsilon \cdot T(\bullet, g, L) \quad \| \tag{4.3.4}$$

holds for every $\varepsilon > 0$, then g is algebraically degenerate.

Proof. We argue by contradiction and assume that the image of g *is* Zariski dense in Y. By [Nog85, Thm. 3.2 on p. 306], there will then exist constants $c_1^+, c_2^+, c_3^+ \in \mathbb{R}^+$ and $c_4 \in \mathbb{R}$ such that an inequality of the form

$$c_1^+ \cdot T(\bullet, g, L) \le N_{\operatorname{Ramification}\rho}(\bullet) + c_2^+ \cdot \varepsilon \cdot \log \bullet + c_3^+ \cdot \log^+ T(\bullet, g, L) + c_4 \quad \| \tag{4.4.1}$$

holds for every number $\varepsilon \in (0,1)$. Choose ε small enough so that

$$(1 + c_2^+ + c_3^+) \cdot \varepsilon < c_1^+$$

[3]Note: We do *not* assume that D_1 has snc support.

and use the assumption that $\mathscr{L}$ is ample to observe

$$\begin{aligned} c_1^+ \cdot T(\bullet, g, L) &\le N_{\mathrm{Ramification}\,\rho}(\bullet) + c_2^+ \cdot \varepsilon \cdot \log \bullet + c_3^+ \cdot \log^+ T(\bullet, g, L) + c_4 && \| \quad (4.4.1) \\ &\le \varepsilon \cdot T(\bullet, g, L) + c_2^+ \cdot \varepsilon \cdot \log \bullet + c_3^+ \cdot \log^+ T(\bullet, g, L) + c_4 && \| \quad (4.3.4) \\ &\le \varepsilon \cdot T(\bullet, g, L) + c_2^+ \cdot \varepsilon \cdot T(\bullet, g, L) + c_3^+ \cdot \log^+ T(\bullet, g, L) + c_4 && \| \quad \text{Lem. 3.7} \\ &\le \varepsilon \cdot T(\bullet, g, L) + c_2^+ \cdot \varepsilon \cdot T(\bullet, g, L) + c_3^+ \cdot \varepsilon \cdot T(\bullet, g, L) + c_4 && \| \quad \text{Lem. 3.7} \\ &= (1 + c_2^+ + c_3^+) \cdot \varepsilon \cdot T(\bullet, g, L). \end{aligned}$$

Given that $T(\bullet, g, L)$ is monotonous and unbounded, this is absurd. □

4.2. **Proof of Theorem 4.1.** Since none of our assumptions refers to L, we may assume without loss of generality that $\mathscr{L}$ is ample. Following [Yam10, proof of Prop. 3.1], we aim to apply Theorem 3.11 ("Lemma on logarithmic derivatives"). To this end, consider the standard Hermitian bundle H of Notation 3.10.

Using Assumption (4.1.4), the counting function for the ramification of ρ is estimated as follows,

$$\begin{aligned} N_{\mathrm{Ramification}\,\rho}(\bullet) &\le (\deg \rho) \cdot N_{1,\mathrm{Ramification}\,\rho}(\bullet) && \forall v \in V : \mathrm{ord}_v \,\mathrm{Ram}.\, \rho \le \deg \rho \\ &\le (\deg \rho) \cdot \sum_{i=1}^{l} N_{\xi_i^* \{0\}}(\bullet) && \text{Ass. (4.1.4)} \end{aligned}$$

Given any $\varepsilon' > 0$, we can give an estimate for each summand,

$$\begin{aligned} N_{\xi_i^* \{0\}}(\bullet) &= T(\bullet, \xi_i, H) - m(\bullet, \xi_i, H, \sigma_0) + O(1) && \text{Thm. 3.8 ("first main")} \\ &\le T(\bullet, \xi_i, H) + O(1) && \text{Lem. 3.5} \\ &= N_{\xi_i^* \{\infty\}}(\bullet) + m(\bullet, \xi_i, H, \sigma_\infty) + O(1) && \text{Thm. 3.8 ("first main")} \\ &= m(\bullet, \xi_i, H, \sigma_\infty) + O(1) && \text{since } \xi_i \text{ is holomorphic} \\ &\le \varepsilon' \cdot T(\bullet, g, L) + O(1) \quad \| && \text{Thm. 3.11 ("log. derivatives")} \end{aligned}$$

Lemma 3.7 will then imply that Inequality (4.3.4) of Noguchi's criterion holds for all $\varepsilon > 0$. The claim thus follows. □

5. C-version of the Bloch-Ochiai theorem, proof of Theorem 1.7

Theorem 1.7 is a direct consequence of the following, stronger Proposition 5.2. The formulation of Proposition 5.2 use the "Albanese of a cover", as introduced and discussed in [KR24a, Sect. 5]. For the reader's convenience, we recall the relevant notions in brief.

Reminder 5.1 (The Albanese of a cover, [KR24a, Def. 5.2 and Prop. 5.5]). Let (X, D) be a C-pair where X is compact Kähler and let $\gamma : \widehat{X} \twoheadrightarrow X$ be a cover. Consider the open sets

$$X^\circ := X \setminus \mathrm{supp}\lfloor D \rfloor \quad \text{and} \quad \widehat{X}^\circ := \gamma^{-1}(X^\circ).$$

Then, there exists a semitoric variety $\mathrm{Alb}(X, D, \gamma)^\circ \subset \mathrm{Alb}(X, D, \gamma)$ and a quasi-algebraic[4] morphism

$$\mathrm{alb}(X, D, \gamma)^\circ : \widehat{X}^\circ \to \mathrm{Alb}(X, D, \gamma)^\circ$$

such that the pull-back morphism for logarithmic reflexive differentials takes its image in the subspace of adapted reflexive differentials. Moreover, if $A^\circ \subset A$ is any semitoric variety, if $a^\circ : \widehat{X}^\circ \to A^\circ$ is any quasi-algebraic morphism whose associated pull-back

[4]Quasi-algebraic = the holomorphic morphism $\mathrm{alb}(X, D, \gamma)^\circ$ extends to a meromorphic map between the compact spaces $\widehat{X}$ and $\mathrm{Alb}(X, D, \gamma)$

morphism for logarithmic reflexive differentials takes its image in the subspace of adapted reflexive differentials, then a factors uniquely as

$$\widehat{X}^\circ \xrightarrow{\operatorname{alb}(X,D,\gamma)^\circ} \operatorname{Alb}(X,D,\gamma)^\circ \xrightarrow{\exists! b^\circ} A^\circ, \qquad a^\circ = b^\circ \circ \operatorname{alb}(X,D,\gamma)^\circ$$

where b° is quasi-algebraic.

Proposition 5.2 (Strong version of Theorem 1.7)**.** *Let (X, D) be a C-pair where X is compact Kähler. If there exists a cover $\gamma : \widehat{X} \twoheadrightarrow X$ such that* $\operatorname{alb}(X, D, \gamma)^\circ$ *is* not *dominant, then every C-entire curve $(\mathbb{C}, 0) \to (X, D)$ is algebraically degenerate.*

Remark 5.3 (Quasi-algebraicity and dominance)*.* The morphism $\operatorname{alb}(X, D, \gamma)^\circ$ is quasi-algebraic and its image is therefore constructible. The word “dominant” in Proposition 5.2 is therefore meaningful.

Remark 5.4 (Relation to Theorem 1.7)*.* In the setting of Theorem 1.7, the assumption $q^+_{\operatorname{Alb}}(X, D) > \dim X$ guarantees the existence of a cover $\gamma : \widehat{X} \twoheadrightarrow X$ such that $\dim \operatorname{Alb}(X, D, \gamma) > \dim X$. In particular, $\operatorname{alb}(X, D, \gamma)^\circ$ is *not* dominant in this setting.

We will prove Proposition 5.2 in the remainder of the present section.

5.1. **Proof of Proposition 5.2.** We argue by contradiction and assume that there exists one C-entire curve $\varphi : (\mathbb{C}, 0) \to (X, D)$ whose image is Zariski dense in X. As before, consider the open sets

$$X^\circ := X \setminus \operatorname{supp}\lfloor D \rfloor \quad \text{and} \quad \widehat{X}^\circ := \gamma^{-1}(X^\circ).$$

The definition of C-morphism guarantees that φ takes its image in $X^\circ \subseteq X$.

Step 1: Galois closure and the Albanese. Functoriality of the Albanese, as spelled out in [KR24a, Lem. 5.8], allows replacing the cover γ by its Galois closure. We will therefore assume without loss of generality that γ is Galois, with group G.

The proof of Proposition 5.2 discusses the Albanese of the cover γ. For brevity of notation, we denote the associated semitoric variety by $\operatorname{Alb}^\circ \subset \operatorname{Alb}$ and write $\operatorname{alb}^\circ : \widehat{X}^\circ \to \operatorname{Alb}^\circ$ for the associated morphism. Recall from [KR24a, Obs. 5.9] that G acts on Alb° by quasi-algebraic automorphisms, in a way that makes the morphism alb° equivariant. Finally, choose an element $\widehat{x} \in \widehat{X}^\circ$ and use its image point

$$0_{\operatorname{Alb}^\circ} := \operatorname{alb}^\circ(\widehat{x}) \in \operatorname{Alb}^\circ$$

to equip Alb° with the structure of a Lie group.

Step 2: Reminder. The assumption that alb° is *not* dominant allows using constructions and results of our earlier paper [KR24a]. For the reader’s convenience, we recall the main points.

Construction of a semitoric quotient variety. In [KR24a, Const. 7-10], we construct a nontrivial semitoric variety $B^\circ \subseteq B$ with G-action, a point $0_{B^\circ} \in B^\circ$ that equips B° with the structure of a Lie group, and a diagram

$$\begin{array}{ccccc} \widehat{X}^\circ & \xrightarrow{\operatorname{alb}^\circ} & \operatorname{Alb}^\circ & \xrightarrow[\beta^\circ,\ \text{group quotient}]{} & B^\circ \\ \Big\downarrow {\scriptstyle \gamma^\circ,\ \text{quotient by } G} & & \Big\downarrow {\scriptstyle \gamma_{\operatorname{Alb}^\circ},\ \text{quotient by } G} & & \Big\downarrow {\scriptstyle \gamma_{B^\circ},\ \text{quotient by } G} \\ X^\circ & \xrightarrow{\delta^\circ} & \operatorname{Alb}^\circ/G & \xrightarrow{\varepsilon^\circ} & B^\circ/G \end{array} \qquad b^\circ = \beta^\circ \circ \operatorname{alb}^\circ \tag{5.5.1}$$

where (among other things) the following holds.

(5.5.2) All horizontal arrows are quasi-algebraic,
(5.5.3) all arrows in the top row are G-equivariant, and
(5.5.4) all arrows in the bottom row are $\mathcal{C}$-morphisms for the $\mathcal{C}$-pairs

$$(X^\circ, D^\circ), \quad \left(\operatorname{Alb}^\circ, 0\right)\big/G, \quad \text{and} \quad \left(B^\circ, 0\right)\big/G.$$

The image of β°. Consider the topological closure $Z := \overline{\operatorname{img} \beta^\circ}$, which is an analytic subset of B because β° is quasi-algebraic. We write $Z^\circ := Z \cap B^\circ$ and set $p := \dim Z$. The following has been shown in [KR24a, Obs. 7.12].

(5.5.5) The variety Z° is positive-dimensional.
(5.5.6) The variety Z° is a proper subset $Z^\circ \subsetneq B^\circ$.

Differentials on B°. Finally, [KR24a, Obs. 7.11] employs methods from Kawamata's proof of the Bloch conjecture, in order to find B°-invariant differentials $\tau_0^\circ, \dots, \tau_p^\circ \in H^0\big(B^\circ,\, \Omega^p_{B^\circ}\big)$ with the following properties.

(5.5.7) The restrictions $\tau_\bullet^\circ|_{Z^\circ_{\mathrm{reg}}}$ are linearly independent top-differentials on Z°_{reg}, and therefore define a $(p+1)$-dimensional linear system $L \subseteq H^0\big(Z^\circ_{\mathrm{reg}}, \omega_{Z^\circ_{\mathrm{reg}}}\big)$.
(5.5.8) The associated meromorphic map $\varphi_L : Z^\circ_{\mathrm{reg}} \dashrightarrow \mathbb{P}^p$ is generically finite.

Step 3: Setup. Let V be the normalized fibre product $\mathbb{C} \times_{X^\circ} \widehat{X}^\circ$, which may be reducible or irreducible. The construction of V extends Diagram (5.5.1) as follows,

$$\begin{array}{ccccccc}
V & \xrightarrow{\widehat{\varphi},\ \text{dense img.}} & \widehat{X}^\circ & \xrightarrow{\operatorname{alb}^\circ} & \operatorname{Alb}^\circ & \xrightarrow{\beta^\circ} & B^\circ \\
\downarrow{\scriptstyle \gamma_V,\ \text{quotient}} & & \downarrow{\scriptstyle \gamma_{\widehat{X}^\circ},\ \text{quotient}} & & \downarrow{\scriptstyle \gamma_{\operatorname{Alb}^\circ},\ \text{quotient}} & & \downarrow{\scriptstyle \gamma_{B^\circ},\ \text{quotient}} \\
\mathbb{C} & \xrightarrow{\varphi,\ \text{dense img.}} & X^\circ & \xrightarrow{\delta^\circ} & \operatorname{Alb}^\circ\big/G & \xrightarrow{\varepsilon^\circ} & B^\circ\big/G.
\end{array}$$

with $\widehat{f} := \beta^\circ \circ \operatorname{alb}^\circ \circ \widehat{\varphi} : V \to B^\circ$ and $f := \varepsilon^\circ \circ \delta^\circ \circ \varphi : \mathbb{C} \to B^\circ\big/G$.

We highlight two elementary facts that will later become relevant.

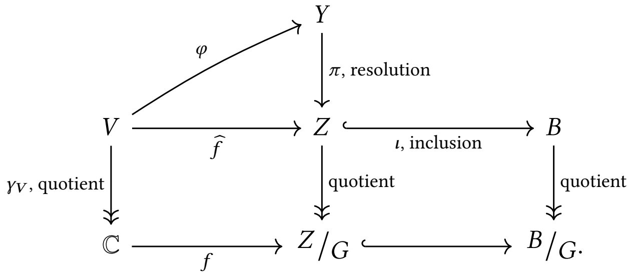
Claim 5.6. The morphism f is a $\mathcal{C}$-morphism from $(\mathbb{C}, 0)$ to the quotient pair $\left(B^\circ, 0\right)\big/G$.

Proof of Claim 5.6. This follows from [KR24b, Prop. 11.1], given that the quotient pair $\big(B^\circ, 0\big)/G$ is uniformizable. □ (Claim 5.6)

Claim 5.7. The natural G-action on V is effective. More precisely: if $g \in G$ is any element, then the fixed point set of the associated translation $V \to V$ is finite.

Proof of Claim 5.7. If $g \in G$ is any hypothetical element whose translation morphism fixes an entire component $V' \subset V$, then equivariance of $\widehat{\varphi}$ implies that the image set $\widehat{\varphi}(V')$ is g-fixed. But that image set is dense in $\widehat{X}^\circ$. □ (Claim 5.7)

Step 4: Resolution of singularities. Consider the semitoric variety $B^\circ \subset B$ and choose a log-resolution $\pi : Y \to Z$ of $(Z, \Delta_B \cap Z)$. Write Δ_Y for the reduced snc divisor on Y whose support equals $\pi^{-1}(\Delta_B)$. As before, write $Y^\circ := Y \setminus \Delta_Y$. The following diagram summarizes the setup,



Remark 5.8. If $V' \subset V$ is any component, then it is clear by construction that $\widehat{f}(V')$ is Zariski-dense in Z. The morphism φ is the canonical lifting of $\widehat{f}$ to the resolution of singularities. This lifting exists because the images $\widehat{f}(V')$ are Zariski-dense and hence not contained in the indeterminacy locus of π^{-1}.

Claim 5.9. The log pair (Y, Δ_Y) is of log-general type.

Proof of Claim 5.9. Recall from [KR24a, Prop. 3.15] that the B°-invariant differentials $\tau^\circ_\bullet \in H^0\big(B^\circ,\, \Omega^p_{B^\circ}\big)$ extend to logarithmic differentials $\tau_\bullet \in H^0\big(B,\, \Omega^p_B(\log \Delta)\big)$. Pulling those back, we obtain sections in $\omega_Y(\log \Delta_Y)$ such that the meromorphic map of the associated linear subsystem of $|K_Y + \Delta_Y|$ is generically finite. □ (Claim 5.9)

Remark 5.10. Claim 5.9 implies that the manifold Y is Moishezon. In particular, there exists a blow-up $\widetilde{Y} \to Y$ where $\widetilde{Y}$ is projective, [Pet94, Cor. 6.10]. Replacing Y by its blow-up, we may assume without loss of generality that the manifold Y is projective.

Claim 5.11. The Albanese morphism $\operatorname{alb}(Y, \Delta_Y)^\circ$ of the log pair (Y, Δ_Y) is generically injective. The dimension of the Albanese satisfies $\dim \operatorname{Alb}(Y, \Delta_Y)^\circ > \dim Y$.

Proof of Claim 5.11. Given that Y° admits a generically injective, quasi-algebraic morphism into B°, generic injectivity of $\operatorname{alb}(Y, \Delta_Y)^\circ$ follows directly from the universal property, as spelled out in [KR24a, Def. 4.2]. For the inequality between the dimension, recall from (5.5.6) that Z° is a proper subset of B°. But [KR24a, Proposition 4.10] implies that Z° generates B° as a group, so that the natural morphism $\operatorname{Alb}(Y, \Delta_Y)^\circ \to B^\circ$ is necessarily surjective. □ (Claim 5.11)

Claim 5.12. The G-action on B° is not free. In particular, there exists a non-trivial, cyclic subgroup $H \subset G$ that acts on B° with a fixed point.

Proof of Claim 5.12. Claim 5.11 allows applying the Logarithmic Bloch-Ochiai Theorem [NW14, Thm. 4.8.17] to the manifold Y and the divisor Δ_Y: entire curves $\mathbb{C} \to Y^\circ$ cannot have Zariski dense images. Together with Remark 5.8 this implies in particular that no component of V is isomorphic to $\mathbb{C}$. The quotient morphism γ_V must therefore be branched, and there do exist group elements $g \in G$ that fix certain points of V. Equivariance of $\widehat{f}$ will then imply that g fixes their images in B°. □ (Claim 5.12)

Step 5: Cyclic subgroups of G and differentials on B. In the situation at hand, where Z° is not contained in the translate of any quasi-algebraic subgroup of B°, the results of Section 2 can be interpreted as an existence statement for differentials with certain factors of automorphy.

Claim 5.13. If $H \subseteq G$ is cyclic and if its action on B° has a fixed point, then there exists a logarithmic differential $\tau_H \in H^0\big(B,\, \Omega^1_B(\log \Delta_B)\big)$ such that the following holds.

(5.13.1) The pull-back differential $\sigma_H := (d\,\widehat{f})\tau_H$ does not vanish identically on any component of V.

(5.13.2) If $h \in H \setminus \{e_H\}$ is any element with associated translation $t_h : V \to V$, then there exists a number $\zeta \in \mathbb{C}^* \setminus \{1\}$ such that $(d\, t_h)\sigma_H = \zeta \cdot \sigma_H$.

Proof of Claim 5.13. We use the notation introduced in Setting 2.1 on page 4. By assumption, the variety Z is not contained in any proper sub-semitorus of Alb°, and then neither are the sets $\widehat{f}(V')$, where $V' \subset V$ is any component. According to Lemma 2.4 on page 4, this implies that none of the restricted morphisms $\widehat{f}|_{V'}$ has its image tangent to the foliation $\mathscr{E}^*_{H,0}$. It follows that there exists a number $\lambda > 0$ and a form $\tau \in E_{H,\lambda}$ such that $(d\,\widehat{f}|_{V'})\tau \neq 0$, for every component $V' \subset V$. But then, Remark 2.2 immediately implies that there exists a number $\lambda > 0$ and a form $\tau_H \in E_{H,\lambda}$ such that (5.13.1) holds. Property (5.13.2) is now an immediate consequence of the description of the H-action on differentials, as given in (2.1.2). □ (Claim 5.13)

Notation and Choice 5.14. Let $\Gamma \subset \mathscr{P}(G)$ be the set of non-trivial, cyclic subgroups of G whose action on B° has at least one fixed point; Claim 5.12 guarantees that this set is not empty. For each of the finitely many $H \in \Gamma$, choose one differential form $\tau_H \in H^0\big(B,\, \Omega^1_B(\log \Delta_B)\big)$ that satisfies the conclusion of Claim 5.13 and write

$$\omega_H := \pi^* \tau_H \qquad \in H^0\big(Y,\, \Omega^1_Y(\log \Delta_Y)\big)$$

$$\sigma_H := g^* \omega_H \qquad \in H^0\big(V,\, \Omega^1_V\big).$$

Following Notation 3.9, we denote the associated meromorphic functions of V as

$$\xi_H := \eta(\sigma_H) \qquad \in H^0\big(V,\, \mathscr{K}_V\big).$$

Maintain this choice for the remainder of the present proof.

Step 6: End of proof. In order to derive a contradiction and to finish the proof of Proposition 5.2, we show that the degeneracy criterion of Theorem 4.1 on page 9 applies to the morphism φ and to the finite collection of differentials, $\{\omega_H : H \in \Gamma\}$. Claims 5.9 and 5.11 together with the following two assertions ensure that the assumptions of Theorem 4.1 are indeed satisfied.

Claim 5.15. For every subgroup $H \in \Gamma$, the meromorphic function ξ_H is holomorphic.

Proof of Claim 5.15. Let $H \in \Gamma$ be any group. To see that ξ_H is holomorphic, recall from Claim 5.6 that f is a C-morphism between $(\mathbb{C}, 0)$ and $(B^\circ, 0)/G$. It will then follow directly from the definition of a "C-morphism" in [KR24b, Def. 8.1] that the differential form $\sigma_H \in H^0\big(V,\, \Omega^1_V\big)$ is a section of the sheaf $\Omega^1_{(\mathbb{C},0,\rho)} = \rho^* \Omega^1_{\mathbb{C}}$. □ (Claim 5.15)

Claim 5.16. For every point $v \in \mathrm{Ramification}\, \rho$, there exists one subgroup $H \in \Gamma$ such that ξ_H vanishes at v.

Proof of Claim 5.16. Given any point $v \in \mathrm{Ramification}\, \gamma_V$, observe that its isotropy group H is non-trivial. Claim 5.7 and the classic statement about "linearization at a fixed point", [HO84, Sect. 1.5], implies that the natural representation morphism

$$G_v \to \mathrm{Gl}(T_V|_v) \cong \mathrm{Gl}(1, \mathbb{C}) \cong \mathbb{C}^*$$

is injective. In particular, H is isomorphic to a subgroup of $\mathbb{C}^*$ and hence cyclic. The fact that $\widehat{f}$ is equivariant implies that $\widehat{f}(v)$ is an H-fixed point of B°. In summary, we find that $H \in \Gamma$. Choose a generator $h \in H$ and recall that there exists a number $\zeta \in \mathbb{C}^* \setminus \{1\}$ such that $(d\, t_h)\sigma_H = \zeta \cdot \sigma_H$. Since $\rho^* dt$ is G-invariant, this implies

$$\xi_H \circ t_h = \zeta \cdot \xi_H.$$

In particular, the function ξ_H must necessarily vanish at the H-fixed point $v \in V$. The claim thus follows. □ (Claim 5.16)

Theorem 4.1 now asserts that the morphism φ is algebraically degenerate, and then so are f_W and f. This contradicts our assumption and ends the proof of Proposition 5.2. □

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