THE ALBANESE OF A C-PAIR

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Abstract. Written with a view towards applications in hyperbolicity, rational points, and entire curves, this paper addresses the problem of constructing Albanese maps within Campana's theory of C-pairs (or "geometric orbifolds"). It introduces C-semitoric pairs as analogues of the (semi)tori used in the classic Albanese theory and follows Serre by defining the Albanese of a C-pair as the universal map to a C-semitoric pairs. The paper shows that the Albanese exists in relevant cases, gives sharp existence criteria, and conjectures that a "weak Albanese" exists unconditionally.

Contents

1. Introduction

This paper constructs Albanese maps for C-pairs, with the goal to provide tools for the study rational points and entire curves in algebraic varieties and complex manifolds. To illustrate our motivation, consider the following two classic theorems of Faltings and Bloch-Ochiai/Kawamata.

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Theorem 1.1 (Faltings' theorem on density of rational points, [Fal91])**.** *Let X be a projective manifold defined over a number field k. If the dimension of its Albanese variety satisfies* $\dim \operatorname{Alb}(X) > \dim X$*, then its rational points are not potentially dense. In other words: If $k \subseteq k'$ is any finite field extension, then k'-rational points are not Zariski dense in X.* □

Theorem 1.2 (Bloch-Ochiai's theorem on entire curves, [Kaw80, Thm. 2])**.** *Let X be a complex projective manifold such that its Albanese variety satisfies* $\dim \operatorname{Alb}(X) > \dim X$*, then entire curves on X are not Zariski dense.* □

Aiming to generalize these results, Campana has formulated a series of far-reaching conjecture that generalize Lang's conjectures and relate potential density of rational points and existence of entire curves to the notion of "specialness" of his theory of C-pairs.

Conjecture 1.3 (Specialness and density of rational points, [Cam11, Conj. 13.21]). Let X be a projective manifold defined over a number field k. Then, X is special if and only if its rational points are potentially dense.

Conjecture 1.4 (Specialness and C-entire curves, [Cam11, Conj. 13.17]). Let X be a complex projective manifold. Then, X is special if and only if X admits a Zariski dense entire curve.

Given that the Albanese appears prominently in Theorems 1.1 and 1.2, we expect that an "Albanese for C-pairs" might play an important role in future progress towards Conjectures 1.3 and 1.4. Section 1.1.4 on page 3 announces first results in this direction.

1.1. **Main results.** The Albanese of a projective manifold X is characterized by universal properties that can be formulated in a number of ways, relating to the geometry or topology of X. Our presentation follows Serre's classic paper [Ser59][1], where the Albanese of a projective manifold X is an Abelian variety $\operatorname{Alb}(X)$ together with a morphism $\operatorname{alb} : X \to \operatorname{Alb}(X)$ such that any other morphism from X to an Abelian variety factors via alb. More generally, we recall in Section 4 that the Albanese of a logarithmic pair (X, D) is a semitoric variety $A^\circ \subset A$, together with a quasi-algebraic morphism $\operatorname{alb} : X \setminus D \to A^\circ$ such that any other quasi-algebraic morphism from $X \setminus D$ to a semitoric variety factors via alb.

1.1.1. *C-semitoric pairs.* For C-pairs (X, D), we argue that the natural analogues of compact tori and semitoric varieties are "C-semitoric pairs", that is, quotients of tori and semitoric varieties, with their natural structure as a quotient C-pair. Section 8 introduces C-semitoric pairs and discusses their main properties. The following non-trivial result suggests that C-semitoric pairs are a geometrically meaningful concept.

Theorem 1.5 (Precise statement in Theorem 8.4)**.** *Quasi-algebraic C-morphisms between C-semitoric pairs come from group morphisms.* □

Following Serre, we define the Albanese of a C-pair (X, D) as a universal, quasi-algebraic C-morphism from (X, D) to a C-semitoric pair.

1.1.2. *The Albanese irregularity.* Given a C-pair (X, D), it turns out that the existence of an Albanese is tied to an invariant of independent interest, the "Albanese irregularity"

$$q^+_{\operatorname{Alb}}(X, D) \in \mathbb{N} \cup \{\infty\}.$$

The Albanese irregularity is bounded from above by the augmented irregularity $q^+(X, D)$, introduced in [KR24a, Sect. 6.1], which measures the dimension of the space of adapted differentials on suitable high covers. The Albanese irregularity differs from the augmented irregularity in that it considers only those adapted differentials that are induced by

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[1] See also the presentation in [Wit08, Appendix A].

morphisms to semitoric varieties. Part II of this paper defines and discusses the Albanese irregularity and the associated "Albanese of a cover" in great details. As one of our major results, we will prove near the end of this paper that special pairs have bounded Albanese irregularity.

Theorem 1.6 (Precise statement in Corollary 7.2 and Remark 7.4)**.** *If* (X, D) *is special in the sense of Campana, then* $q^+_{\mathrm{Alb}}(X, D) \le \dim X$. □

In spite of the notion's obvious importance, we do not fully understand the geometric meaning of the (potentially strict) inequality $q^+(X, D) \le q^+_{\mathrm{Alb}}(X, D)$. Section 10.2 gathers a number of open questions.

1.1.3. *The Albanese of a C-pair.* With all preparations in place, the main result of our paper is now formulated as follows.

Theorem 1.7 (Precise statement in Theorem 9.2 and Proposition 9.5)**.** *Let* (X, D) *be a nc C-pair, where X is a compact Kähler manifold. Then, the following statements are equivalent.*

(1.7.1) An Albanese of the C-pair (X, D) *exists.*
(1.7.2) The Albanese irregularity is finite, $q^+_{\mathrm{Alb}}(X, D) < \infty$. □

We speculate that if $q^+_{\mathrm{Alb}}(X, D) = \infty$, it might still make sense to define an Albanese, either with a weaker universal property, or in the broader setup of ind-varieties. We refer to Section 10.1 for a discussion.

1.1.4. *Preview: Pairs with high irregularity.* In the forthcoming paper [KR24b], we develop the beginnings of a Nevanlinna theory for C-pairs, with the goal to study hyperbolicity properties of pairs with high irregularity. A first application generalizes the classic Bloch-Ochiai Theorem 1.2 to the setting of C-pairs: If $q^+_{\mathrm{Alb}}(X, D) > \dim X$, then every C-entire curve $(\mathbb{C}, 0) \to (X, D)$ is algebraically degenerate. This theorem explicitly includes the case where $q^+_{\mathrm{Alb}}(X, D) = \infty$. It establishes Conjecture 1.4 for some non-special varieties.

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Part I. Preparation

2. Notation and standard facts

2.1. **Global conventions.** This paper works in the category of complex analytic spaces, though all the material in this paper will work in the complex-algebraic setting, often with less involved definitions and proofs. With very few exceptions, we follow the notation of the standard reference texts [GR84, Dem12, NW14]. An *analytic variety* is a reduced, irreducible complex space. For clarity, we refer to holomorphic maps between analytic varieties as *morphisms* and reserve the word *map* for meromorphic mappings.

We use the language of C-pairs, as surveyed in [KR24a], and freely refer to definitions and results from [KR24a] throughout the present text. The reader might wish to keep a hardcopy within reach.

2.2. **Quasi-algebraic morphisms.** Let X and Y be normal analytic varieties. In contrast to the algebraic setting, it is generally *not* possible to extend a morphism between Zariski open subsets to a meromorphic map between X and Y: the exponential map does not extend to a meromorphic map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. Morphisms that do extend meromorphically will be of special interest. Following [NW14], we refer to them as *quasi-algebraic*.

Definition 2.1 (Quasi-algebraic morphism)**.** *Let (X, D_X) and (Y, D_Y) be pairs where X and Y are compact. A morphism between the open parts, $X^\circ \to Y^\circ$, is called* quasi-algebraic with respect to the compactifications X and Y *if it extends to a meromorphic map $X \dashrightarrow Y$.*

Notation 2.2 (Quasi-algebraic morphisms to $\mathbb{C}$ and $\mathbb{C}^*$). Recall that $\mathbb{C}$ and $\mathbb{C}^*$ admit a unique normal compactification to $\mathbb{P}^1$. If (X, D_X) is a pair where X is compact, it is therefore meaningful to say that a morphism $X^\circ \to \mathbb{C}$ or $X^\circ \to \mathbb{C}^*$ is quasi-algebraic. Analogously, it makes sense to say that a function in $\mathscr{O}_X(X^\circ)$ or in $\mathscr{O}^*_X(X^\circ)$ is quasi-algebraic.

Definition 2.3 (Family of quasi-algebraic morphisms)**.** *In the setting of Definition 2.1, let Z be any normal analytic variety. A* family of quasi-algebraic morphisms over Z *is a morphism $X^\circ \times Z \to Y^\circ$ that extends to a meromorphic map $X \times Z \dashrightarrow Y$.*

For lack of an adequate reference, we include proofs of the following elementary facts.

Lemma 2.4 (Elementary properties)**.** *Let (X, D_X), (Y, D_Y) and (Z, D_Z) be pairs, where X, Y and Z are compact. Assume that a sequence of morphism is given,*

$$\begin{tikzcd}
X^\circ \arrow[r, "\alpha^\circ"'] \arrow[rr, bend left, "\gamma^\circ"] & Y^\circ \arrow[r, "\beta^\circ"'] & Z^\circ,
\end{tikzcd}$$

where α° is quasi-algebraic. Then, the following holds.

(2.4.1) *If β° is quasi-algebraic, then γ° is quasi-algebraic.*
(2.4.2) *If α° is dominant and γ° is quasi-algebraic, then β° is quasi-algebraic.*

Proof. Only (2.4.2) will be shown. Replacing X and Y by suitable bimeromorphic models, we may assume that there exists a commutative diagram as follows,

$$\begin{tikzcd}
X \arrow[r, two heads, "{\alpha,\text{ surjective}}"'] \arrow[rr, bend left, "\gamma"] & Y \arrow[r, dotted, "\exists ?\, \beta"'] & Z \\
X^\circ \arrow[u, hook] \arrow[r, "{\alpha^\circ,\text{ dominant}}"] \arrow[rr, bend right, "\gamma^\circ"'] & Y^\circ \arrow[u, hook] \arrow[r, "\beta^\circ"] & Z^\circ. \arrow[u, hook]
\end{tikzcd}$$

The image $\Gamma \subset Y \times Z$ of the product morphism $\alpha \times \gamma : X \to Y \times Z$ is analytic by the proper mapping theorem. Commutativity of the diagram guarantees that Γ is bimeromorphic to Y, and hence the graph of the desired meromorphic map $\beta : Y \dashrightarrow Z$. □

Quasi-algebraic morphisms to $\mathbb{C}^*$ enjoy the following strong rigidity property.

Lemma 2.5 (Families of quasi-algebraic morphisms to $\mathbb{C}^*$)**.** *Let (X, D_X) be a pair where X is compact, let Z be any normal analytic variety and let $\varphi^\circ : X^\circ \times Z \to \mathbb{C}^*$ be a family of quasi-algebraic morphisms over Z. Then, there exist functions $f^\circ \in \mathscr{O}^*_X(X^\circ)$ and $g \in \mathscr{O}^*_Z(Z)$ such that the equality $\varphi^\circ(x, z) = f^\circ(x) \cdot g(z)$ holds for every $(x, z) \in X^\circ \times Z$.*

Proof. Extend φ° to a meromorphic map $\varphi : X \times Z \to \mathbb{P}^1$ and view φ as a meromorphic function. Choosing a point $z_0 \in Z$, we would like to compare φ to the meromorphic function $F(x, z) := \varphi(x, z_0)$. For that, consider the associated principal divisors, $\operatorname{div} \varphi$ and $\operatorname{div} F$ in $\operatorname{Div}(X \times Z)$. Both divisors are supported on $(X \setminus X^\circ) \times Z$ and are hence of product form. Their restrictions to $X \times \{z_0\}$ agree. It follows that the two divisors are equal, so that $G := \varphi / F$ is a holomorphic function on $X \times Z$ without zeros or poles. The function G

is constant on the (compact!) fibres of the projection map $X \times Z \to Z$ and hence descends to a function $g \in \mathscr{O}^*_Z(Z)$ on the normal space Z. To conclude, set $f^\circ(x) := \varphi^\circ(x, z_0)$. ☐

3. Semitoric varieties, quasi-algebraic morphisms and groups

The Albanese of a compact Kähler manifold is a compact complex torus. We will recall in Section 4 that the Albanese of a logarithmic pair is a more complicated object: a semitorus together with a preferred bimeromorphic equivalence class of a compactification. For the reader's convenience, we recall the relevant notions and prove a number of elementary statements that are not readily found in the literature.

We follow conventions and the language of the textbook [NW14] and refer the reader to [NW14, Sect. 4 and 5] for details, proofs and references to the original literature.

Definition 3.1 (Semitorus, presentation, [NW14, Def. 5.1.5 and Sect. 5.1.5])**.** *A* semitorus *is a connected commutative complex Lie group A° that admits a surjective Lie group morphism $\pi^\circ : A^\circ \twoheadrightarrow T$, where T is a compact complex torus and $\ker \pi^\circ \cong (\mathbb{C}^*)^{\times \bullet}$. Lie group morphisms of this form are called* presentations *of the semitorus A°.*

Remark 3.2. Semitori also appear under the name *quasi-tori* in the literature, [Kob98, p. 119]. Presentations are not unique. A given semitorus might allow two different presentations whose associated compact complex tori are hugely different.

3.1. **Semitoric varieties.**

Semitoric varieties are the analytic analogues of Abelian varieties, complex tori and toric varieties. The following definition is taken almost verbatim from [NW14].

Definition 3.3 (Semitoric variety, [NW14, Def. 5.3.3])**.** *A* semitoric variety *is a compact analytic variety A together with a holomorphic action $\aleph^\circ \circlearrowleft A$ of a semitorus $\aleph^\circ$ such that the following holds.*

(3.3.1) There is a dense open orbit $A^\circ \subset A$ on which $\aleph^\circ$ acts freely.

(3.3.2) There exists a presentation $\pi^\circ : \aleph^\circ \twoheadrightarrow T$ with the following properties.

- *Using (3.3.1) to identify A° with $\aleph^\circ$, the morphism π° extends to an $\aleph^\circ$-equivariant morphism $\pi : A \twoheadrightarrow T$.*
- *For every point $t \in T$ the fibre $A_t = \pi^{-1}(t)$ is isomorphic to a smooth toric variety. In other words, A_t admits the structure of a smooth algebraic variety such that the action of $\ker \pi^\circ$ on A_t is algebraic.*

Explanation 3.4 (First item of (3.3.2)). Extendability of π° in the first item of (3.3.2) is independent of the identification. The extension is unique if it exists.

Explanation 3.5 (Second item of (3.3.2)). Spelled out in detail, the second item of (3.3.2) requires that A_t admits the structure of a smooth algebraic variety such that the action of $\ker \pi^\circ$ on A_t is algebraic. Item (3.3.2) immediately implies that A is smooth and $A \setminus A^\circ$ is an snc divisor.

Warning 3.6. An identification of A° with a Lie group is not part of the data that defines a semitoric variety.

Notation 3.7 (Semitoric varieties). For brevity of notation, we will frequently write "Let A° be a semitorus, and let $A^\circ \subset A$ be a semitoric compactification…" to say that A is a compactification where the action on A° on itself by left multiplication extends to A in a way that makes A with the action $A^\circ \circlearrowleft A$ a semitoric variety.

Notation 3.8 (Semitoric varieties). For brevity of notation, we will frequently write "Let $A^\circ \subset A$ be a semitoric variety…" to say that we consider a compact analytic variety A and a dense open subset $A^\circ \subset A$ where A° is biholomorphic to a semitorus $\aleph^\circ$ that acts holomorphically on A such that

- the subset $A^\circ \subset A$ is an orbit on which $\aleph^\circ$ acts freely, and
- the analytic variety A together with the action of $\aleph^\circ$ is a semitoric variety in the sense of Definition 3.3.

Notation 3.9 (Semitoric varieties as logarithmic pairs). Given a semitoric variety $A^\circ \subset A$, we will often consider the associated logarithmic pair (A, Δ) and write $\Omega^p_A(\log \Delta)$, with the implicit understanding that $\Delta := A \setminus A^\circ$ is the difference divisor. If there is more than one semitoric variety involved in the discussion, we write (A, Δ_A) for clarity.

Notation 3.10 (Quasi-algebraic morphisms). Given two semitoric varieties, $A^\circ \subset A$ and $B^\circ \subset B$, we follow Definition 2.1 and say that a morphism $A^\circ \to B^\circ$ is quasi-algebraic if it extends to a meromorphic map $A \dashrightarrow B$. Along similar lines, if (X, D) is any pair where X is compact, it makes sense to say that morphisms between the open parts, $A^\circ \to X^\circ$ and $X^\circ \to A^\circ$, are quasi-algebraic.

3.2. **Elementary properties.** For later reference, we state several facts about quasi-algebraic morphisms between semitoric varieties. The proofs are tedious but mostly elementary, and left to the reader. In fancy words, Facts 3.11–3.13 can be seen to give an equivalence of categories between presentations of semitori and bimeromorphic equivalence classes of semitoric compactifications.

Fact 3.11 (Uniqueness of presentation)**.** *The presentation of $\aleph^\circ$ in Definition 3.3 is unique. More precisely, there exists a unique presentation $\pi^\circ : \aleph^\circ = A^\circ \twoheadrightarrow T$ that extends to an $\aleph^\circ$-equivariant fibre bundle $\pi : A \twoheadrightarrow T$.* □

Fact 3.12 (Existence for given presentation, [NW14, Thm. 5.1.35])**.** *Let A° be a semitorus and let $\pi^\circ : A^\circ \twoheadrightarrow T$ be a presentation. If F is any smooth toric variety compactifying $F^\circ := (\pi^\circ)^{-1}(0_T)$, then there exists a semitoric variety $A^\circ \subset A$ with associated morphism $\pi : A \to T$ where $\pi^{-1}(0_T)$ is isomorphic to F as an F°-space.* □

A group morphism between the open parts of semitoric varieties is quasi-algebraic if and only if it respects the associated presentations. In particular, we find that the bimeromorphic equivalence class of a semitoric compactification is uniquely determined by the presentation.

Fact 3.13 (Quasi-algebraic group morphisms and presentations)**.** *Let A° and B° be semitori, and let $A^\circ \subset A$ and $B^\circ \subset B$ be semitoric compactifications, with associated morphisms $\pi_A : A \twoheadrightarrow T_A$ and $\pi_B : B \twoheadrightarrow T_B$. If $\sigma^\circ : A^\circ \to B^\circ$ is any holomorphic group morphism, then the following two statements are equivalent.*

(3.13.1) The morphism σ° is quasi-algebraic.

(3.13.2) There exists a holomorphic group morphism $\tau : T_A \to T_B$, where $\tau \circ \pi_A = \pi_B \circ \sigma^\circ$. □

Proof of the implication (3.13.1) ⇒ (3.13.2). Lemma 2.4 guarantees that the composed map $\pi^\circ_B \circ \sigma^\circ$ is quasi-algebraic. Since the compact complex torus B does not contain rational curves, we find that the quasi-algebraic morphism $\pi^\circ_B \circ \sigma^\circ$ factors via the $(\mathbb{C}^*)^{\times\bullet}$-fibre bundle π°_A. We obtain a morphism $f_T : T_A \to T_B$ and a commutative diagram as follows,

$$\begin{tikzcd}
A^\circ \arrow[r, "\sigma^\circ"] \arrow[d, two heads, "\pi^\circ_A"'] & B^\circ \arrow[d, two heads, "\pi^\circ_B"] \\
T_A \arrow[r, "\tau"'] & T_B.
\end{tikzcd} \tag{3.13.3}$$

The morphism τ maps 0_{T_A} to 0_{T_B} and is hence a group morphism, [NW14, Def. 5.1.36]. □

If the equivalent conditions of Fact 3.13 hold, there is a little more that we can say: σ° extends to a morphism between A and B if and only its restriction to the central fibre extends to a morphism.

Fact 3.14 (Morphisms and bimeromorphic maps)**.** *In the setting of Fact 3.13, assume that σ° is quasi-algebraic, with associated meromorphic map $\sigma : A \dashrightarrow B$. Then, the following two statements are equivalent.*

(3.14.1) The meromorphic map σ is a morphism.

(3.14.2) The meromorphic map $\sigma|_{\pi_A^{-1}(0_{T_A})} : \pi_A^{-1}(0_{T_A}) \dashrightarrow \pi_B^{-1}(0_{T_B})$ is a morphism. □

On semitoric varieties, a differential form is logarithmic if and only if it is invariant.

Proposition 3.15 (Invariant differentials and logarithmic differentials)**.** *In the setting of Definition 3.3, the following statements hold for every number $p \in \mathbb{N}$.*

(3.15.1) The locally free sheaf $\Omega^p_A(\log \Delta)$ is free.

(3.15.2) Every A°-invariant differential form $\tau^\circ \in H^0\big(A^\circ,\, \Omega^p_{A^\circ}\big)$ extends to a logarithmic form $\tau \in H^0\big(A,\, \Omega^p_A(\log \Delta)\big)$.

(3.15.3) Every logarithmic form in $H^0\big(A,\, \Omega^p_A(\log \Delta)\big)$ is A°-invariant.

Proof. Item (3.15.1) is [NW14, Cor. 5.4.5]. For Item (3.15.2), observe that every A°-invariant differential form $\tau^\circ \in H^0\big(A^\circ,\, \Omega^p_{A^\circ}\big)$ can be written as a sum of wedge products of 1-differentials. To prove Item (3.15.2), it will therefore suffice to consider the case $p = 1$. The group A° acts on itself by left multiplication. By assumption, this actions extends to an action of A° on A that stabilizes Δ. The A°-invariant vector fields on A° that are induced by this action will therefore extend to sections of $\mathscr{T}_A(-\log \Delta)$. Using (3.15.1), the case $p = 1$ of Item (3.15.2) now follows by taking duals.

Item (3.15.3) follows from (3.15.1) and (3.15.2), given that the dimensions of the spaces $H^0\big(A^\circ,\, \Omega^p_{A^\circ}\big)^{A^\circ}$ and $H^0\big(A, \Omega^p_A(\log \Delta)\big)$ agree. □

Remark 3.16 (Pull-back of logarithmic differentials I). Given a semitoric variety $A^\circ \subset A$ and a nc log pair (X, D), we are often interested in quasi-algebraic morphisms $a^\circ : X^\circ \to A^\circ$. Given that X and A are smooth and that a is holomorphic away from a small subset of X, there exists a pull-back morphism for logarithmic differentials

$$\mathrm{d}\, a : H^0\big(A, \Omega^1_A(\log \Delta)\big) \to H^0\big(X,\, \Omega^1_X(\log D)\big)$$

that restricts on X° to the standard pull-back $\mathrm{d}\, a^\circ$.

Remark 3.17 (Pull-back of logarithmic differentials II). Generalizing Remark 3.16, given a semitoric variety $A^\circ \subset A$, a log pair (X, D) that is not necessarily nc, and a quasi-algebraic morphism $a^\circ : X^\circ \to A^\circ$, there exists a pull-back morphism for logarithmic differentials

$$\mathrm{d}\, a : H^0\big(A, \Omega^1_A(\log \Delta)\big) \to H^0\big(X,\, \Omega^{[1]}_X(\log D)\big)$$

that restricts on X°_{reg} to the standard pull-back $\mathrm{d}\, a^\circ$.

3.3. **Quasi-algebraic morphisms.** In contrast to the algebraic setting, a morphism between semitori need not be a group morphism, even if it respects the neutral elements of the group structure. For an example, consider the morphism $\mathbb{C}^* \to \mathbb{C}^*$, $t \mapsto \exp(t-1)$. The situation improves for quasi-algebraic morphisms of semitoric varieties.

Proposition 3.18 (Quasi-algebraic morphisms and group morphisms)**.** *Let A° and B° be semitori, and let $A^\circ \subset A$ and $B^\circ \subset B$ be semitoric compactifications. Let $f^\circ : A^\circ \to B^\circ$ be any quasi-algebraic morphism of analytic varieties. If $f^\circ(0_{A^\circ}) = 0_{B^\circ}$, then f° is a morphism of complex Lie groups.*

Proof. In order to prepare for the proof, consider the associated presentations $\pi^\circ_A : A^\circ \twoheadrightarrow T_A$ and $\pi^\circ_B : B^\circ \twoheadrightarrow T_B$. Fact 3.13 equips us with a group morphism $f_T : T_A \to T_B$ forming a commutative diagram as follows,

$$\begin{tikzcd} A^\circ \arrow[r, "f^\circ"] \arrow[d, two heads, "\pi_A^\circ"'] & B^\circ \arrow[d, two heads, "\pi_B^\circ"] \\ T_A \arrow[r, "f_T"'] & T_B. \end{tikzcd} \tag{3.18.1}$$

We would like to show that f° is a group morphism. For this, consider the auxiliary morphism

$$\xi^\circ : A^\circ \times A^\circ \to B^\circ, \quad (x, y) \mapsto f^\circ(x) + f^\circ(y) - f^\circ(x + y).$$

To conclude, we need to show that $\xi^\circ \equiv 0_{B^\circ}$ or equivalently that ξ° is constant. The assumption that f° is quasi-algebraic and [NW14, Prop. 5.3.5] together guarantee that ξ° extends to a meromorphic map $\xi : A \times A \dashrightarrow B$ and is hence quasi-algebraic. The following property follows from the assumption that $f^\circ(0_{A^\circ}) = 0_{B^\circ}$.

$$\forall a \in A^\circ : \xi^\circ(a, 0_{A^\circ}) = \xi^\circ(0_{A^\circ}, a) = 0_{B^\circ} \tag{3.18.2}$$

There is more that we can say. If $(x, y) \in A^\circ \times A^\circ$ is any pair of points, then

$$\begin{aligned} (\pi_B^\circ \circ \xi^\circ)(x, y) &= \pi_B^\circ\big(f^\circ(x) + f^\circ(y) - f^\circ(x + y)\big) && \text{definition} \\ &= (\pi_B^\circ \circ f^\circ)(x) + (\pi_B^\circ \circ f^\circ)(y) - (\pi_B^\circ \circ f^\circ)(x + y) && \pi_B^\circ \text{ a grp. morph.} \\ &= (f_T \circ \pi_A^\circ)(x) + (f_T \circ \pi_A^\circ)(y) - (f_T \circ \pi_A^\circ)(x + y) && \text{Diagram (3.18.1)} \\ &= 0_{T_B} && f_T \circ \pi_A^\circ \text{ a grp. morph.} \end{aligned}$$

In summary, we find that ξ° takes its image in $(\pi_B^\circ)^{-1}(0_{T_B})$. Fixing one identification $(\pi_B^\circ)^{-1}(0_{T_B}) \cong (\mathbb{C}^*)^{\times\bullet}$, Lemma 2.5 allows writing ξ° in product form. More precisely, there exist functions $a_\bullet, b_\bullet \in \mathscr{O}_A^*(A^\circ)$ such that

$$\xi^\circ(x, y) = \big(a_1(x) \cdot b_1(y), \dots, a_n(x) \cdot b_n(y)\big), \quad \text{for every } (x, y) \in A^\circ \times A^\circ.$$

Equation (3.18.2) will then imply that ξ° is constant. □

Corollary 3.19 (Quasi-algebraic morphisms between open parts of semitoric varieties). *Let $A^\circ \subset A$ and $B^\circ \subset B$ be semitoric varieties and let $f^\circ : A^\circ \to B^\circ$ be a quasi-algebraic morphism of analytic varieties. Then, the following holds.*

(3.19.1) The fibres of f° are of pure dimension.
(3.19.2) Any two non-empty fibres of f° are of the same dimension.
(3.19.3) If f° is quasi-finite, then it is finite.
(3.19.4) If f° is finite and surjective, then it is étale. □

Corollary 3.20 (Quasi-algebraic automorphisms). *Let $A^\circ \subset A$ be a semitoric variety. Once we choose an element $0_{A^\circ} \in A^\circ$ to equip A° with the structure of a holomorphic Lie group, the group of quasi-algebraic automorphisms of the analytic variety A° decomposes as a semidirect product $(\text{translations}) \rtimes (\text{group morphisms})$.* □

Corollary 3.21 (Semitoric compactification with additional symmetry). *Let $A^\circ \subset A_1$ be a semitoric variety, and let $G \subset \operatorname{Aut}(A^\circ)$ be a finite group of quasi-algebraic automorphisms. Then, there exists a semitoric variety $A^\circ \subset A_2$, such that the following holds.*

- *The analytic varieties A_1 and A_2 are bimeromorphic.*
- *The G-action on A° extends equivariantly to A_2.*

Proof. As before, write $\pi^\circ : A^\circ \twoheadrightarrow T$ for the unique presentation that extends to an A°-equivariant morphism $\pi : A_1 \twoheadrightarrow T$. Corollary 3.20 allows assuming without loss of generality that G is a finite group of quasi-algebraic group morphisms. Fact 3.13 will then guarantee that G acts by group morphisms on T in a way that makes the morphism π° equivariant. In particular, the G-action fixes the point 0_T and stabilizes the fibre $F^\circ :=$

$(\pi^\circ)^{-1}(0_T) \cong (\mathbb{C}^*)^{\times\bullet}$. Toric geometry will then allow choosing[2] a G-equivariant toric compactification $F^\circ \subset F$, and Fact 3.12 presents us with a semitoric compactification $A^\circ \subset A_2$, fibred over T with typical fibre F. Fact 3.13 ensures that A_1 and A_2 are bimeromorphic, and Fact 3.14 asserts that the G-action on A° extends equivariantly to A_2. □

3.4. **Quasi-algebraic subgroups.** In analogy to the notion of a quasi-algebraic morphism, a quasi-algebraic subgroup of a semitorus is a subgroup that extends to an analytic set in a preferred compactification. A full discussion of this notion is found in [NW14, Sect. 5.3.4].

Definition 3.22 (Quasi-algebraic subgroup, [NW14, Def. 5.3.14])**.** *Let A° be a semitorus, and let $A^\circ \subset A$ be a semitoric compactification. An analytic subgroup $H^\circ \subset A^\circ$ is called* quasi-algebraic for the semitoric compactification $A^\circ \subset A$ *if the topological closure of H° in A is an analytic subset.*

3.4.1. *Elementary properties.* We state two facts about quasi-algebraic subgroups for later reference. The elementary proofs are left to the reader.

Fact 3.23 (Quasi-algebraic subgroups are semitori, [NW14, Prop. 5.3.13])**.** *In the setting of Definition 3.22, quasi-algebraic subgroups are again semitori.* □

Warning 3.24 (Analytic subgroups need not be semitori). Despite claims to the contrary in the literature, cf. [Kob98, Lem. 3.8.18], closed analytic subgroups of semitori need not be semitori in general. See [NW14, Ex. 5.1.44] and the references there for an example.

The following fact implies that the notion of "quasi-algebraic subgroup" depends only on the bimeromorphic equivalence class of a semitoric compactification.

Fact 3.25 (Dependence on choice of compactification)**.** *Let A° be a semitorus, and let $A^\circ \subset A_1$ and $A^\circ \subset A_2$ be two bimeromorphic semitoric compactifications. Then, a subgroup $H^\circ \subset A^\circ$ is quasi-algebraic for the semitoric compactification $A^\circ \subset A_1$ if and only if it is quasi-algebraic for the semitoric compactification $A^\circ \subset A_2$.* □

3.4.2. *Lattice structure.* As usual in algebra, quasi-algebraic subgroups form a complete lattice. We refrain from going into any details here and state the only fact that will be relevant for us later.

Fact 3.26 (Existence of a smallest group)**.** *In the setting of Definition 3.22, the intersection of arbitrarily many quasi-algebraic subgroups is quasi-algebraic. In particular, given any subset $I \subset A^\circ$, there exists a unique smallest quasi-algebraic subgroup that contains I.* □

3.4.3. *Quotients.* Semitoric varieties are stable under quotients by quasi-algebraic groups, in the following sense.

Fact 3.27 (Existence of a quotients, [NW14, Thm. 5.3.13])**.** *Let A° be a semitorus, and let $A^\circ \subset A$ be a semitoric compactification. If $H^\circ \subset A^\circ$ is a quasi-algebraic subgroup, then the quotient $Q^\circ := A^\circ/H^\circ$ is a semitorus and there exists a semitoric compactification $Q^\circ \subset Q$ that renders the quotient morphism $q^\circ : A^\circ \twoheadrightarrow Q^\circ$ quasi-algebraic.* □

3.4.4. *Examples.* Throughout this paper, quasi-algebraic subgroups appear as kernels of quasi-algebraic group morphisms and as fixed point sets of quasi-algebraic group action. We recall the relevant facts.

Fact 3.28 (Kernels of quasi-algebraic group morphisms)**.** *Let A° and B° be a semitori, and let $A^\circ \subset A$ and $B^\circ \subset B$ be semitoric compactifications. If $\alpha^\circ : A^\circ \to B^\circ$ is any quasi-algebraic morphism of complex Lie groups, then $\ker(\alpha^\circ) \subset A^\circ$ is quasi-algebraic for the semitoric compactification $A^\circ \subset A$.* □

---

[2] Since G is finite, every fan can be refined to become stable under the action of G on $N_{\mathbb{R}}$.

Proposition 3.29 (Fixed points of quasi-algebraic groups actions)**.** *Let A° be a semitorus, and let $A^\circ \subset A$ be a semitoric compactification. Let $G \subset \operatorname{Aut}(A^\circ)$ be a finite group that acts on A° by quasi-algebraic automorphisms. If*

$$X \subset \{\vec{a} \in A^\circ \;:\; \text{isotropy } G_{\vec{a}} \text{ is not trivial}\}$$

is any irreducible complex subspace, then X is contained in the translate of a proper quasi-algebraic subgroup of A°.

Proof. We assume that the group G is non-trivial, or else there is nothing to prove. Since G is finite, there will be an element $g \in G \setminus \{e\}$ that fixes X pointwise. Shrinking G and enlarging X, we may therefore assume without loss of generality that G is cyclic, $G = \langle g \rangle$, and that X is a component of $\operatorname{Fix}(G)$.

Recall from Proposition 3.18 that the action of g on A° is of the form

$$g : A^\circ \to A^\circ, \quad \vec{a} \mapsto \varphi^\circ(\vec{a}) - \vec{a}_0$$

where $\varphi^\circ : A^\circ \to A^\circ$ is a quasi-algebraic group morphism and $\vec{a}_0 \in A^\circ$ is a constant. It follows that $\vec{a} \in \operatorname{Fix}(g)$ if and only if $\bigl(\varphi^\circ - \operatorname{Id}_{A^\circ}\bigr)(\vec{a}) = \vec{a}_0$. If $\vec{x} \in X$ is any element, this implies that

$$\operatorname{Fix}(G) = \ker(\varphi^\circ - \operatorname{Id}_{A^\circ}) + \vec{x}.$$

But by Fact 3.28, the components of $\ker(\varphi^\circ - \operatorname{Id}_{A^\circ})$ are translates of quasi-algebraic subgroups. □

4. THE ALBANESE OF A LOGARITHMIC PAIR

To prepare for the slightly involved constructions later in this paper, we recall a number of facts about the Albanese for logarithmic pairs. For lack of references, we include a full discussion the Albanese construction in the singular Kähler case. We refer the reader to [Ser59] and [Wit08, Appendix A] for general results in the algebraic setting, and to [NW14, Sect. 4.5] for a construction of the Albanese for logarithmic pairs (X, D) in case where X is a compact Kähler manifold and D a reduced divisor that does not necessarily have snc support.

Setting 4.1. Let (X, D) be a log pair where X is compact. In line with [KR24a, Notation 2.14], denote the open part by $X^\circ := X \setminus D$.

Definition 4.2 (The Albanese for compact log pairs)**.** *Assume Setting 4.1. An Albanese of (X, D) is a semitoric variety $\operatorname{Alb}(X, D)^\circ \subset \operatorname{Alb}(X, D)$ together with a quasi-algebraic morphism*

$$\operatorname{alb}(X, D)^\circ : X^\circ \to \operatorname{Alb}(X, D)^\circ$$

that satisfies the following universal property. If $A^\circ \subset A$ is any semitoric variety and

$$a^\circ : X^\circ \to A^\circ$$

is any quasi-algebraic morphism, then there exists a unique morphism b° that makes the following diagram commute,

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"'] \arrow[rr, bend left, "a^\circ"] & \operatorname{Alb}(X, D)^\circ \arrow[r, "\exists ! b^\circ"'] & A^\circ. \end{tikzcd} \tag{4.2.1}$$

The morphism b° is quasi-algebraic.

Remark 4.3 (Quasi-Albanese). The Albanese of an snc logarithmic pair also appears under the name "quasi-Albanese" in the literature, cf. [Fuj24].

Notation 4.4 (Empty boundary). If the divisor D in Definition 4.2 is zero, we will often drop it from the notation and write $\operatorname{alb}(X)^\circ : X^\circ \to \operatorname{Alb}(X)^\circ = \operatorname{Alb}(X)$ for brevity.

Remark 4.5 (Compactification and presentation of $\operatorname{Alb}(X,D)^\circ$). In the setting of Definition 4.2, recall from Facts 3.11 and 3.13 that the semitoric compactification $\operatorname{Alb}(X,D)^\circ \subset \operatorname{Alb}(X,D)$ defines a unique presentation of the semitorus $\operatorname{Alb}(X,D)^\circ$. If (X,D) is snc, the construction presented in Section 4.2 will show that this presentation equals the natural morphism $\operatorname{Alb}(X,D)^\circ \twoheadrightarrow \operatorname{Alb}(X)$ induced by the universal property.

Explanation 4.6. The reader coming from algebraic geometry might wonder why Definition 4.2 is so complicated. The reason is this: if V° is a smooth, quasi-projective variety and if $V^\circ \subset V_1$ and $V^\circ \subset V_2$ are two projective compactifications, then V_1 and V_2 are birational and there exists a third compactification that dominates both.

This is no longer true in complex geometry, where two compactifications need not necessarily be bimeromorphic, and where the bimeromorphic equivalence class of a particular compactification is often part of the data. Along these lines, the Albanese is not just the semitorus $\operatorname{Alb}(X,D)^\circ$, but the semitorus together with a bimeromorphic equivalence class of a compactification $\operatorname{Alb}(X,D)$. The word “quasi-algebraic” that appears all over Definition 4.2 ensures that all morphisms respect the classes of the compactifications.

4.1. **Uniqueness.**

The universal property of the Albanese guarantees that $\operatorname{Alb}(X,D)^\circ$ and $\operatorname{alb}(X,D)^\circ$ are unique up to unique isomorphism. The left-invariant compactification $\operatorname{Alb}(X,D)$ is bimeromorphically unique. Following the classics, we abuse notation and refer to any Albanese as “the Albanese”, with associated semitoric *Albanese variety* $\operatorname{Alb}(X,D)^\circ \subset \operatorname{Alb}(X,D)$ and *Albanese morphism* $\operatorname{alb}(X,D)^\circ$. Once we fix a point $0_{\operatorname{Alb}(X,D)^\circ} \in \operatorname{Alb}(X,D)^\circ$ to equip $\operatorname{Alb}(X,D)^\circ$ with the structure of a Lie group, Fact 3.25 on page 8 allows talking about subgroups of $\operatorname{Alb}(X,D)^\circ$ that are quasi-algebraic for the semitoric compactification $\operatorname{Alb}(X,D)^\circ \subset \operatorname{Alb}(X,D)$.

4.2. **Existence.**

The existence of an Albanese is well-known for snc pairs, but hardly discussed in the literature for arbitrary Kähler pairs. We briefly recall the arguments in the snc setting, use resolutions of singularities to construct a candidate for the Albanese in general and prove that this candidate satisfies the properties spelled out in Definition 4.2 above.

Proposition 4.7 (Existence of the Albanese of a Kähler log pair)**.** *In Setting 4.1, assume that X is Kähler. Then, an Albanese of (X,D) exists.*

We begin the proof by recalling the classic construction for snc pairs. For singular pairs, Construction 4.8 will show how to build an Albanese using a resolution of singularities. We conclude the proof of Proposition 4.7 on the following page, showing that Construction 4.8 does indeed satisfy the necessary universal property.

Proof of Proposition 4.7 is (X,D) is snc. If the pair (X,D) of Setting 4.1 is snc, choose a point $x \in X^\circ$ and consider the group morphism

$$i : \pi_1(X^\circ, x) \to H^0\big(X,\ \Omega^1_X(\log D)\big)^*$$

obtained by path integration. Set

$$\operatorname{Alb}(X,D)^\circ := H^0\big(X,\ \Omega^1_X(\log D)\big)^* \Big/ \operatorname{img}(i)$$

and define $\operatorname{alb}(X,D)^\circ$ by path integration. Hodge theory guarantees that $\operatorname{Alb}(X,D)^\circ$ is a semitorus. It admits a presentation as a principal $(\mathbb{C}^*)^{\times\bullet}$-bundle over $\operatorname{Alb}(X)$, and hence by Fact 3.12 on page 5 an equivariant compactification $\operatorname{Alb}(X,D)$ as a $(\mathbb{P}^1)^{\times\bullet}$-bundle over $\operatorname{Alb}(X)$. A local computation shows that $\operatorname{alb}(X,D)^\circ$ is quasi-algebraic for this compactification. More precisely, it extends to a meromorphic map $X \dashrightarrow \operatorname{Alb}(X,D)$ that is holomorphic on the big open subset $X \setminus (\operatorname{supp} D)_{\mathrm{sing}}$. We refer the reader to [NW14, Sect. 4] for details and proofs. □

Construction 4.8 (Construction of the Albanese of a log pair). Assume the setting of Proposition 4.7. For the reader's convenience, we subdivided the construction into relatively independent steps.

Step 1 in Construction 4.8, Resolution of singularities. Choose a log-resolution $\pi : \widetilde{X} \twoheadrightarrow X$, consider the reduced snc divisor $\widetilde{D} := \operatorname{supp} \pi^{-1}(D)$ on $\widetilde{X}$ and write $\widetilde{X}^\circ := \widetilde{X} \setminus \widetilde{D}$. The proof of Proposition 4.7 in the snc case provides us with an Albanese of $(\widetilde{X}, \widetilde{D})$ that we briefly denote as

(4.8.1)
$$\begin{tikzcd} \widetilde{X} \arrow[d, two heads, "{\pi,\ \text{log resolution}}"'] & \supseteq & \widetilde{X}^\circ \arrow[d, two heads, "\pi^\circ"'] \arrow[r, "{\widetilde{a}^\circ := \operatorname{alb}(\widetilde{X},\widetilde{D})^\circ}", "\text{quasi-algebraic}"'] & \underbrace{\operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ}_{=:\widetilde{A}^\circ} & \subseteq & \underbrace{\operatorname{Alb}(\widetilde{X}, \widetilde{D})}_{=:\widetilde{A}} \\ X & \supseteq & X^\circ. \end{tikzcd}$$

Step 2 in Construction 4.8, Quotients by subgroups of $\widetilde{A}^\circ$. Choose an element $0_{\widetilde{A}^\circ} \in \widetilde{A}^\circ$ in order to equip $\widetilde{A}^\circ$ with the structure of a Lie group. If $H^\circ \subseteq \widetilde{A}^\circ$ is any quasi-algebraic subgroup, recall from Fact 3.27 that the quotient

$$A^\circ_{H^\circ} := \widetilde{A}^\circ \big/ H^\circ$$

is a semitorus and there exists a semitoric compactification $A^\circ_{H^\circ} \subset A_{H^\circ}$ that renders the quotient morphism $q^\circ_{H^\circ} : \widetilde{A}^\circ \twoheadrightarrow A^\circ_{H^\circ}$ quasi-algebraic. If the composed map

$$q^\circ_{H^\circ} \circ \widetilde{a}^\circ : \widetilde{X}^\circ \to A^\circ_H$$

is constant on π°-fibers, then it factors via π°, and we obtain an extension of Diagram (4.8.1) as follows,

(4.8.2)
$$\begin{tikzcd} \widetilde{X} \arrow[d, two heads, "{\pi,\ \text{log resolution}}"'] & \supseteq & \widetilde{X}^\circ \arrow[d, two heads, "\pi^\circ"'] \arrow[rr, "{\widetilde{a}^\circ,\ \text{quasi-algebraic}}"] & & \widetilde{A}^\circ \arrow[d, two heads, "q^\circ_{H^\circ}"] & \subseteq & \widetilde{A} \arrow[d, dashed, two heads, "q_{H^\circ}"] \\ X & \supseteq & X^\circ \arrow[rr, "a^\circ_{H^\circ}"'] & & A^\circ_{H^\circ} & \subseteq & A_{H^\circ} \end{tikzcd}$$

The quotient carries a natural structure of a Lie group that makes $q^\circ_{H^\circ}$ a group morphism. Lemma 2.4 guarantees that $a^\circ_{H^\circ}$ is again quasi-algebraic.

Step 3 in Construction 4.8, Identifying a suitable subgroup of A°. Aiming to construct an Albanese for (X, D) using the construction of Step 2, we need to find a quasi-algebraic subgroup $H^\circ \subseteq A^\circ$ to which Step 2 can be applied. To this end, consider the set of all subgroups that satisfy the assumptions of Step 2,

$$\mathcal{H}^\circ := \{B^\circ \subseteq \widetilde{A}^\circ \text{ quasi-algebraic } : q^\circ_{B^\circ} \circ \widetilde{a}^\circ \text{ is constant on } \pi^\circ\text{-fibers}\}.$$

Take H° as the infimum of $\mathcal{H}^\circ$ in the complete lattice of all quasi-algebraic subgroups $\widetilde{A}^\circ$. In other words, define

$$H^\circ := \bigcap_{B^\circ \in \mathcal{H}^\circ} B^\circ$$

and recall from Fact 3.26 that H° is indeed a quasi-algebraic subgroup. With this choice, observe that $q^\circ_{H^\circ} \circ \widetilde{a}^\circ$ is again constant on π°-fibers, so that H° is in fact the minimal element of $\mathcal{H}^\circ$. Step 2 equips us with a semitoric compactification $A^\circ_{H^\circ} \subset A_{H^\circ}$ and a diagram of Form (4.8.2). Write $\operatorname{Alb}(X, D)^\circ \subset \operatorname{Alb}(X, D)$ for $A^\circ_{H^\circ} \subseteq A_{H^\circ}$ and denote the quasi-algebraic morphism $a^\circ_{H^\circ}$ by

$$\operatorname{alb}(X, D)^\circ : X^\circ \to \operatorname{Alb}(X, D)^\circ.$$

Construction 4.8 ends here.

Proof of Proposition 4.7. It remains to show that the varieties and morphism of Construction 4.8 satisfy the conditions spelled out in Definition 4.2 above. To this end, assume that $A^\circ \subset A$ is a semitoric variety and $a^\circ : X^\circ \to A^\circ$ is a quasi-algebraic morphism. Item (2.4.1) of Lemma 2.4 guarantees that $a^\circ \circ \pi^\circ : \widetilde{X}^\circ \to A^\circ$ is quasi-algebraic. The universal property of the Albanese $\operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ$ of the snc pair (X, D) thus gives us a unique quasi-algebraic morphism $\widetilde{b}^\circ$ of Lie groups that makes the following diagram commute,

$$\begin{tikzcd}
\widetilde{X}^\circ \arrow[r, "\operatorname{alb}(\widetilde{X},\widetilde{D})^\circ"] \arrow[d, two heads, "\pi^\circ"'] & \operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ \arrow[d, two heads, "q^\circ_{H^\circ}"] \arrow[dr, "\exists ! \widetilde{b}^\circ"] & \\
X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"] \arrow[rr, bend right, "a^\circ"'] & \operatorname{Alb}(X, D)^\circ \arrow[r, dotted, "\text{want: } b^\circ"] & A^\circ.
\end{tikzcd}$$

Consider the element $\widetilde{b}^\circ(0_{\operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ}) \in A^\circ$ to equip A° with the structure of a Lie group that makes $\widetilde{b}^\circ$ a group morphism. Since the composed map

$$\widetilde{b}^\circ \circ \operatorname{alb}(\widetilde{X},\widetilde{D})^\circ = \operatorname{alb}(X, D)^\circ \circ \pi^\circ$$

is constant on π°-fibres, the choice of H° in Step 3 of Construction 4.8 immediately guarantees that

$$\ker q^\circ_{H^\circ} = H^\circ \subseteq \ker \widetilde{b}^\circ.$$

It follows that there is a unique Lie group morphism $b^\circ : \operatorname{Alb}(X, D)^\circ \to A^\circ$ that makes the diagram commute. Item (2.4.2) of Lemma 2.4 guarantees that b° is quasi-algebraic, as desired. The fact that $\widetilde{b}^\circ$ is unique as a morphism of varieties implies that b° is unique as a morphism of varieties. □

4.3. **Additional properties.** The Albanese has numerous properties that we will use in the sequel. While all of those necessarily follow from the universal property that determines the Albanese uniquely, we find it often easier to use the concrete construction of the Albanese in 4.8, which quickly reduces us to the snc setting where all results are known and readily citable.

Proposition 4.10 (Image of alb generates Alb). *In Setting 4.1, assume that X is Kähler. Let $x \in X^\circ$ be any point and use*

$$0_{\operatorname{Alb}(X,D)^\circ(x)} := \operatorname{alb}(X, D)^\circ(x) \in \operatorname{Alb}(X, D)^\circ$$

to equip $\operatorname{Alb}(X, D)^\circ$ *with the structure of a Lie group. Then, the image of* $\operatorname{alb}(X, D)^\circ$ *generates* $\operatorname{Alb}(X, D)^\circ$ *as an Abelian group.*

Proof. If (X, D) is snc, this is [NW14, Prop. 4.5.11]. In general, consider Diagram (4.8.2) of Construction 4.8, use that $\operatorname{img} \operatorname{alb}(\widetilde{X}, \widetilde{D})^\circ$ generates $\operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ$ and that the quotient map

$$q^\circ_{H^\circ} : \operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ \twoheadrightarrow \operatorname{Alb}(X, D)^\circ$$

is surjective. □

Proposition 4.11 (Group actions). *In Setting 4.1, assume that X is Kähler. Given a finite subgroup G of* $\operatorname{Aut}(X, D)$*, there exists an Albanese* $\operatorname{Alb}(X, D)^\circ \subset \operatorname{Alb}(X, D)$ *where G acts on the pair* $(\operatorname{Alb}(X, D), \Delta_{\operatorname{Alb}(X,D)^\circ})$ *in a way that makes the morphisms*

$$X^\circ \xrightarrow{\operatorname{alb}(X,D)^\circ} \operatorname{Alb}(X, D)^\circ \hookrightarrow \operatorname{Alb}(X, D)$$

equivariant.

Remark 4.12 (Equivariant presentation of the Albanese). In the setting of Proposition 4.11, the group G will also act on the pair $(X, 0)$ and hence on the Albanese $\operatorname{Alb}(X)$. Continuing Remark 4.5, we leave it to the reader to check that the presentation morphism

$$\operatorname{Alb}(X, D) \longrightarrow \operatorname{Alb}(X)$$

is likewise G-equivariant.

Proof of Proposition 4.11. The group action on $\operatorname{Alb}(X, D)^\circ$ are of course induced by the universal property. In fact, given any automorphism $g \in \operatorname{Aut}(X, D)$, consider the diagram

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"] \arrow[d, "g"'] & \operatorname{Alb}(X, D)^\circ \arrow[d, "\exists!\sigma(g)"] \\ X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"'] & \operatorname{Alb}(X, D)^\circ, \end{tikzcd}$$

where $\sigma(g)$ is the quasi-algebraic morphism of semitori given by the universal property. An elementary computation shows that the morphism

$$\operatorname{Aut}(X, D) \to \operatorname{Aut}\big(\operatorname{Alb}(X, D)^\circ\big), \quad g \mapsto \sigma(g)$$

is indeed a group morphism that makes the morphism to $\operatorname{Alb}(X)$ equivariant. Corollary 3.21 on page 8 allows finding a G-equivariant, semitoric compactification. □

4.3.1. *Resolution of singularities.* Construction 4.8 makes it easy to compare the Albanese of a pair with the Albanese of a resolution of singularities. To begin, we observe that a surjection of pairs induces a surjection between the Albanese varieties.

Observation 4.13 (Surjective morphisms). Let (X, D_X) and (Y, D_Y) be two log pairs, where X and Y are compact Kähler spaces. Given a quasi-algebraic surjection $\varphi^\circ : X^\circ \twoheadrightarrow Y^\circ$, the universal property of the Albanese yields a diagram of the form

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D_X)^\circ"] \arrow[d, two heads, "\varphi^\circ"'] & \operatorname{Alb}(X, D_X)^\circ \arrow[d, "\operatorname{alb}(\varphi^\circ)"] \\ Y^\circ \arrow[r, "\operatorname{alb}(Y,D_Y)^\circ"'] & \operatorname{Alb}(Y, D_Y)^\circ. \end{tikzcd}$$

Choosing a point $x \in X^\circ$ and using

$$\begin{aligned} 0_{\operatorname{Alb}(X,D_X)^\circ} &:= \operatorname{alb}(X,D_X)^\circ(x) && \in \operatorname{Alb}(X, D_X)^\circ \\ 0_{\operatorname{Alb}(Y,D_Y)^\circ} &:= \operatorname{alb}(Y,D_Y)^\circ(\varphi^\circ x) && \in \operatorname{Alb}(Y, D_Y)^\circ \end{aligned}$$

to equip $\operatorname{Alb}(X, D_X)^\circ$ and $\operatorname{Alb}(Y, D_Y)^\circ$ with Lie group structures, $\operatorname{alb}(\varphi^\circ)$ becomes a quasi-algebraic Lie group morphism. The image of $\operatorname{alb}(\varphi^\circ)$ is thus a subgroup that contains the image of $\operatorname{alb}(Y, D_Y)^\circ$ and hence generates $\operatorname{Alb}(Y, D_Y)^\circ$ as a group. It follows that $\operatorname{alb}(\varphi^\circ)$ is surjective.

Proposition 4.14 (The Albanese and the Albanese of a log resolution)**.** *In Setting 4.1, assume that X is Kähler. Let $\pi : \widetilde{X} \to X$ be a log resolution of the pair (X, D). Consider the reduced divisor $\widetilde{D} :=$ supp $\pi^{-1}(D)$ and the associated diagram*

$$\begin{tikzcd} \widetilde{X}^\circ \arrow[r, "\operatorname{alb}(\widetilde{X},\widetilde{D})^\circ"] \arrow[d, two heads, "\pi^\circ"'] & \operatorname{Alb}(\widetilde{X}, \widetilde{D})^\circ \arrow[d, two heads, "\operatorname{alb}(\pi^\circ),\ \textit{surj. by Obs. 4.13}"] \\ X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"'] & \operatorname{Alb}(X, D)^\circ. \end{tikzcd}$$

In particular, observe that

$$\dim \operatorname{Alb}(X,D)^\circ \le \dim \operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ. \tag{4.14.1}$$

If X° has only rational singularities, then $\operatorname{alb}(\pi^\circ)$ *is isomorphic and Inequality* (4.14.1) *is an equality.*

Proof. The assumption that X° has only rational singularities implies that every form $\sigma \in H^0\big(\widetilde{X}^\circ,\, \Omega^1_{\widetilde{X}}\big)$ vanishes when restricted to the smooth locus of any π°-fibre, [Nam01, Lem. 1.2]. This applies in particular to differential forms coming from $\operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ$. Since the cotangent bundle of $\operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ$ is free, we find that $\operatorname{alb}(\widetilde{X},\widetilde{D})^\circ$ maps π°-fibres to points. The map $\operatorname{alb}(\widetilde{X},\widetilde{D})^\circ$ therefore factors via π°, and the group H° of Construction 4.8 is therefore trivial, $H^\circ = \{0\}$. □

4.3.2. *Description in terms of differentials.* As in the classic case, the Albanese of a singular pair can be described in terms of differentials, as a Lie group quotient of a dualized space of one-forms. The following observation makes this statement precise.

Observation 4.15 (Presentation of the Albanese as a Lie group quotient). In the setting of Proposition 4.14, choose a point $\widetilde{x} \in \widetilde{X}$ and use

$$\begin{aligned} 0_{\operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ} &:= \operatorname{alb}(\widetilde{X},\widetilde{D})^\circ(\widetilde{x}) && \in \operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ \\ 0_{\operatorname{Alb}(X,D)^\circ} &:= \operatorname{alb}(X,D)^\circ(\pi^\circ\widetilde{x}) && \in \operatorname{Alb}(X,D)^\circ \end{aligned}$$

to equip $\operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ$ and $\operatorname{Alb}(X,D)^\circ$ with Lie group structures that make $\operatorname{alb}(\pi^\circ)$ a group morphism. Since π is surjective, the push-forward of any torsion free sheaf is torsion free, and we obtain an injection

$$\pi_*\Omega^1_{\widetilde{X}}(\log \widetilde{D}) \hookrightarrow \Omega^{[1]}_X(\log D), \tag{4.15.1}$$

which presents $\operatorname{Alb}(X,D)^\circ$ as a Lie group quotient,

$$\begin{aligned} &H^0\big(X,\, \Omega^{[1]}_X(\log D)\big)^* \twoheadrightarrow H^0\big(\widetilde{X},\, \Omega^1_{\widetilde{X}}(\log \widetilde{D})\big)^* && \text{dual of (4.15.1)} && (4.15.2)\\ &\qquad\qquad \twoheadrightarrow \operatorname{Alb}(\widetilde{X},\widetilde{D})^\circ && \text{quotient by } \pi_1(\widetilde{X}^\circ,\widetilde{x}) && (4.15.3)\\ &\qquad\qquad \twoheadrightarrow \operatorname{Alb}(X,D)^\circ && \text{quotient by quasi-algebraic.} && (4.15.4) \end{aligned}$$

The pull-back morphism for logarithmic differentials introduced in Remark 3.17 on page 7,

$$d\operatorname{alb}(X,D) : H^0\Big(\Omega^1_{\operatorname{Alb}(X,D)}(\log \Delta)\Big) \to H^0\big(X,\, \Omega^{[1]}_X(\log D)\big),$$

is the induced map between dual Lie algebras, hence injective. Observation 4.15 ends here.

Corollary 4.16 (Dimension of Alb)**.** *In Setting 4.1, assume that X is Kähler. Then, the dimension of* $\operatorname{Alb}(X,D)^\circ$ *satisfies the inequality*

$$\dim \operatorname{Alb}(X,D)^\circ \le h^0\big(X,\, \Omega^{[1]}_X(\log D)\big). \tag{4.16.1}$$

If the pair (X,D) is Du Bois and if X° has rational singularities, then (4.16.1) *is an equality.*

Proof. The inequality follows directly from Observation 4.15 above. Assuming that (X,D) is Du Bois and that X° has rational singularities, we show that the composed surjection (4.15.2)–(4.15.4) has a discrete kernel.

To begin, recall that since X° has rational singularities, Proposition 4.14 asserts that (4.15.4) is an isomorphism. Its kernel is hence trivial. The kernel of (4.15.3) is discrete. We claim that (4.15.2) is likewise isomorphic. To this end, decompose (4.15.1) as

$$\pi_*\Omega^1_{\widetilde{X}}(\log \widetilde{D}) \overset{a}{\hookrightarrow} \pi_*\Omega^1_{\widetilde{X}}(\log \widetilde{D} + \operatorname{Exc}\pi) \overset{b}{\hookrightarrow} \Omega^{[1]}_X(\log D). \tag{4.16.2}$$

Recall from [KS21, Cor. 1.8, Rem. 1.9] that a is isomorphic because X° has rational singularities. Recall from [GK14, Thm. 4.1] that b is isomorphic because (X,D) is Du Bois. □

Remark 4.17 (Relation to Minimal Model Theory). Recall the classic results that log-canonical pairs are Du Bois and that the space underlying a log-terminal pair has rational singularities. Corollary 4.16 will therefore give an equality if the pair (X, D) is dlt in the sense of Minimal Model Theory, [KM98, Def. 2.37].

Remark 4.18 (Improvements). Corollary 4.16 is probably not optimal. Using the notion of "weakly rational singularities" introduced in [KS21, Sect. 1.4] and the extension results of [Par23, Tig23], the assumptions on rational singularities might be weakened, at the expense of introducing technically challenging singularity classes, [KM98, Thm. 5.23] and [Kol13, Sect. 6.2].

We leave the proof of the following fact to the reader.

Fact 4.19 (Image of $d\,a$ and $\ker(b)$). *In the setting of Observation 4.15, assume we are given a factorization as in Diagram (4.2.1). Consider the linear subspace*

$$W := \operatorname{img}\Big(d\,a : H^0\big(A, \Omega^1_A(\log \Delta_A)\big) \to H^0\big(X, \Omega^{[1]}_X(\log D)\big)\Big),$$

write $W^\perp \subseteq H^0\big(X, \Omega^{[1]}_X(\log D)\big)^*$ *for its annihilator and recall from Observation 4.15 above that there exists a natural surjection of Lie groups*

$$\eta : H^0\big(X, \Omega^{[1]}_X(\log D)\big)^* \twoheadrightarrow \operatorname{Alb}(X, D)^\circ.$$

Then, $\ker(b^\circ) = \eta\big(W^\perp\big)$. □

4.4. **Examples.** The following example shows that the Inequalities (4.14.1) and (4.16.1) will generally be strict, even for pairs with no boundary and with the simplest log-canonical singularities.

Example 4.20 (Strict inequalities). Consider closed immersions $E \subsetneq \mathbb{P}^2 \subsetneq \mathbb{P}^3$ where E is an elliptic curve and where $\mathbb{P}^2$ is linearly embedded into $\mathbb{P}^3$. Let $X \subset \mathbb{P}^3$ be the projective cone over E. Since X is rationally connected, morphisms to semitori will necessarily be constant. It follows that the Albanese of $(X, 0)$ will be trivial. Next, let $\pi : \widetilde{X} \to X$ be the resolution of singularities, obtained as the blow-up of the unique singular point in X. Since $\widetilde{X}$ is a $\mathbb{P}^1$-bundle over E, its Albanese equals E. The following diagram summarizes the situation,

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\operatorname{alb}(\widetilde{X})"] \arrow[d, two heads, "\pi"'] & \operatorname{Alb}(\widetilde{X}) \arrow[d, two heads, "\operatorname{alb}(\pi)"] \arrow[r, phantom, "="] & E \\
X \arrow[r, "\operatorname{alb}(X)"'] & \operatorname{Alb}(X) \arrow[r, phantom, "="] & \{0\}.
\end{tikzcd}$$

Inequality (4.14.1) is strict in this case. The inequalities

$$1 = h^0\big(E, \Omega^1_E\big) \le h^0\big(\widetilde{X}, \Omega^1_{\widetilde{X}}\big) = h^0\big(\widetilde{X}, \pi_*\Omega^1_{\widetilde{X}}\big) \le h^0\big(X, \Omega^{[1]}_X\big)$$

show that (4.16.1) is likewise strict.

In case where the underlying space X of a pair (X, D) is smooth, the following example shows that the Albanese of (X, D) agrees with the Albanese of a log resolution. Together with [NW14, Rem. 4.5.10], this implies that our construction of Albanese agrees with that of [NW14, Sect. 4.5], even though the two constructions might initially look different.

Example 4.21 (Albanese in case where the underlying space is smooth). In Setting 4.1, assume that X is a Kähler manifold and let $\operatorname{Alb}(X, D)^\circ \subset \operatorname{Alb}(X, D)$ be an Albanese of (X, D), with Albanese morphism $\operatorname{alb}(X, D)^\circ : X^\circ \to \operatorname{Alb}(X, D)^\circ$.

If $\pi : \widetilde{X} \twoheadrightarrow X$, is a log-resolution of the pair (X, D) and $\widetilde{D} := \operatorname{supp} \pi^{-1}(D)$ the reduced preimage divisor, then Construction 4.8 immediately shows that $\operatorname{Alb}(X, D)^\circ \subset \operatorname{Alb}(X, D)$ is also an Albanese of $(\widetilde{X}, \widetilde{D})$, with Albanese morphism $\operatorname{alb}^\circ(\widetilde{X}, \widetilde{D}) = \operatorname{alb}(X, D)^\circ \circ \pi$.

Part II. The Albanese of a cover

5. The Albanese of a cover and the Albanese irregularity

Generalizing the Albanese of a logarithmic pair, we construct an Albanese attached to every cover $\widehat{X} \twoheadrightarrow X$ of a given C-pair (X, D), which need not be logarithmic. Recalling that the Albanese of a logarithmic snc pair is a "universal" morphism to a semitoric variety that induces all logarithmic differentials, we define the Albanese of a cover as a "universal" morphism from $\widehat{X}$ to a semitoric variety such that every pull-back differential is adapted. We consider the following setting throughout the present section.

Setting 5.1. Let (X, D) be a C-pair where X is compact and let $\gamma : \widehat{X} \twoheadrightarrow X$ be a cover of (X, D). Consider the reduced divisor

$$\widehat{D} := \left(\gamma^* \lfloor D \rfloor\right)_{\mathrm{red}} \in \operatorname{Div}(\widehat{X})$$

and write $\widehat{X}^\circ := \widehat{X} \setminus \operatorname{supp} \widehat{D}$.

We underline that Setting 5.1 does *not* assume that γ is adapted, that $\widehat{X}$ is smooth, or that $\gamma^* D$ has nc support. The following definition of the Albanese will therefore use adapted *reflexive* differentials.

Definition 5.2 (The Albanese of a cover of a C-pair)**.** *Assume Setting 5.1. An* Albanese of (X, D, γ) *is a semitoric variety* $\operatorname{Alb}(X, D, \gamma)^\circ \subset \operatorname{Alb}(X, D, \gamma)$ *together with a quasi-algebraic morphism*

$$\operatorname{alb}(X, D, \gamma)^\circ : \widehat{X}^\circ \to \operatorname{Alb}(X, D, \gamma)^\circ$$

such that the following holds.

(5.2.1) The pull-back morphism for logarithmic differentials of Remark 3.17,

$$H^0\Big(\Omega^1_{\operatorname{Alb}(X,D,\gamma)}(\log \Delta)\Big) \xrightarrow{d\,\operatorname{alb}(X,D,\gamma)} H^0\Big(\widehat{X},\ \Omega^{[1]}_{\widehat{X}}(\log \widehat{D})\Big),$$

takes its image in the subspace $H^0\Big(\widehat{X},\ \Omega^{[1]}_{(X,D,\gamma)}\Big) \subseteq H^0\Big(\widehat{X},\ \Omega^{[1]}_{\widehat{X}}(\log \widehat{D})\Big)$.

(5.2.2) If $A^\circ \subset A$ is any semitoric variety, if $a^\circ : \widehat{X}^\circ \to A^\circ$ is any quasi-algebraic morphism such that the pull-back morphism

$$d\,a : H^0\big(A,\ \Omega^1_A(\log \Delta)\big) \to H^0\Big(\widehat{X},\ \Omega^{[1]}_{\widehat{X}}(\log \widehat{D})\Big)$$

takes its image in $H^0\Big(\widehat{X},\ \Omega^{[1]}_{(X,D,\gamma)}\Big)$*, then a factors uniquely as*

$$\begin{tikzcd}\widehat{X}^\circ \arrow[r, "\operatorname{alb}(X,D,\gamma)^\circ"'] \arrow[rr, bend left, "a^\circ"] & \operatorname{Alb}(X, D, \gamma)^\circ \arrow[r, "\exists! b^\circ"'] & A^\circ,\end{tikzcd}$$

where b° is quasi-algebraic.

Remark 5.3 (Pull-back of p-differentials). Item (5.2.1) of Definition 5.2 can be phrased in terms of sheaf morphisms. Recall from [NW14, Cor. 5.4.5] that the locally free sheaf $\Omega^1_{\operatorname{Alb}(X,D,\gamma)}(\log \Delta)$ is free and hence globally generated. Item (5.2.1) is therefore equivalent to the following, seemingly stronger statement: If p is any number, then the composed pull-back morphism

$$\left(\operatorname{alb}(X, D, \gamma)^\circ\right)^* \Omega^p_{\operatorname{Alb}(X,D,\gamma)^\circ} \to \Omega^{[p]}_{\widehat{X}^\circ}$$

takes its image in the subsheaf $\Omega^{[p]}_{(X^\circ,D^\circ,\gamma)} \subseteq \Omega^{[p]}_{\widehat{X}^\circ}$.

5.1. **The Albanese irregularity.** Given a $\mathcal{C}$-pair (X, D) and a cover $\gamma : \widehat{X} \twoheadrightarrow X$, the dimension of the Albanese is an important invariant of the triple (X, D, γ).

Definition 5.4 (Albanese irregularity, augmented Albanese irregularity)**.** *Assume Setting 5.1. If an Albanese exists, then refer to the number*

$$q_{\mathrm{Alb}}(X, D, \gamma) := \dim \mathrm{Alb}(X, D, \gamma)^\circ$$

as the Albanese irregularity of (X, D, γ). *The number*

$$q^+_{\mathrm{Alb}}(X, D) = \sup\big\{ q_{\mathrm{Alb}}(X, D, \gamma) \mid \gamma \textit{ a cover} \big\} \in \mathbb{N} \cup \{\infty\}$$

is the augmented Albanese irregularity of the $\mathcal{C}$-pair (X, D).

We will show in Section 7 that the augmented Albanese irregularity $q^+_{\mathrm{Alb}}(X, D)$ is finite if X is Kähler and if the $\mathcal{C}$-pair (X, D) is special.

5.2. **Uniqueness and existence.** As before, the universal property spelled out in Item (5.2.2) implies that $\mathrm{Alb}(X, D, \gamma)^\circ$ is unique up to unique isomorphism. The compactification $\mathrm{Alb}(X, D, \gamma)$ is bimeromorphically unique. As before, we abuse notation and refer to any Albanese as "the Albanese", with associated semitoric *Albanese variety* $\mathrm{Alb}(X, D, \gamma)^\circ \subset \mathrm{Alb}(X, D, \gamma)$ and quasi-algebraic *Albanese morphism* $\mathrm{alb}(X, D, \gamma)^\circ$.

Proposition 5.5 (Existence of the Albanese of a cover)**.** *In Setting 5.1, assume that X is Kähler. Then, an Albanese of (X, D, γ) exists. Its dimension satisfies the inequality*

$$\dim \mathrm{Alb}(X, D, \gamma)^\circ \le q(X, D, \gamma).$$

If $\widehat{x} \in \widehat{X}^\circ$ is any point and if we use

$$0_{\mathrm{Alb}(X,D,\gamma)^\circ} := \mathrm{alb}(X, D, \gamma)^\circ(\widehat{x}) \in \mathrm{Alb}(X, D, \gamma)^\circ$$

to equip $\mathrm{Alb}(X, D, \gamma)^\circ$ with the structure of a Lie group, then the image of $\mathrm{alb}(X, D, \gamma)^\circ$ generates $\mathrm{Alb}(X, D, \gamma)^\circ$ as an Abelian group.

The proof of Proposition 5.5 requires some preparation. We give it in Section 6.2, starting from Page 23 below. Assuming for the moment that the Albanese can be shown to exist, the subsequent Sections 5.3–5.5 gather its most important properties.

5.3. **Inequalities between irregularities.** If (X, D) is a $\mathcal{C}$-pair where X is compact Kähler and if $\gamma : \widehat{X} \twoheadrightarrow X$ is a cover of (X, D), we have seen in Proposition 5.5 that the Albanese irregularity is bounded by the irregularity,

$$q_{\mathrm{Alb}}(X, D, \gamma) \le q(X, D, \gamma). \tag{5.6.1}$$

There are settings where Inequality (5.6.1) is strict and the natural morphism

$$H^0\Big(\Omega^1_{\mathrm{Alb}(X,D,\gamma)}(\log \Delta)\Big) \xrightarrow{\mathrm{d\,alb}(X,D,\gamma)} H^0\Big(\widehat{X},\ \Omega^{[1]}_{(X,D,\gamma)}\Big)$$

is not surjective. Equivalently said: there are settings where it is not true that every adapted reflexive differential on $\widehat{X}$ comes from a logarithmic differential on the Albanese. A first example has already been discussed in the previous section.

Example 5.7 (Strict inequality between irregularities). Let X be the cone over the elliptic curve discussed in Example 4.20 on page 15. Let $\mathrm{Id}_X : X \longrightarrow X$ be the trivial covering. We have seen in Example 4.20 that

$$q(X, 0, \mathrm{Id}_X) = h^0\big(X,\ \Omega^1_{(X,0,\mathrm{Id}_X)}\big) = h^0\big(X,\ \Omega^{[1]}_X\big)$$

is positive while $\mathrm{Alb}(X, 0, \mathrm{Id}_X) = \mathrm{Alb}(X)$ is a point, so that $q_{\mathrm{Alb}}(X, 0, \mathrm{Id}_X) = 0$.

Example 5.7 might seem artificial, given that X has an elliptic singularity. While smooth examples exist, they are more complicated to construct. We have therefore chosen to publish details elsewhere. Section 10.2 asks related questions for inequalities between augmented irregularities.

5.4. **Functoriality in sequences of covers.** The following immediate consequence of the universal property will be used later.

Lemma 5.8 (Functoriality of the Albanese). *Let (X, D) be a C-pair where X is compact Kähler. Let*

$$\widehat{X}_1 \xrightarrow{\gamma_1} \widehat{X}_2 \xrightarrow{\gamma_2} X$$

be a sequence of covers. Consider the reduced divisors

$$\widehat{D}_2 := \big(\gamma_2^* \lfloor D \rfloor\big)_{\text{red}} \quad \textit{and} \quad \widehat{D}_1 := \big((\gamma_2 \circ \gamma_1)^* \lfloor D \rfloor\big)_{\text{red}}$$

and write $\widehat{X}_\bullet^\circ := \widehat{X}_\bullet \setminus \operatorname{supp} \widehat{D}_\bullet$. Then, there exists a unique surjection c° that renders the following diagram commutative,

$$\begin{tikzcd}
\widehat{X}_1^\circ \arrow[r, "\operatorname{alb}(X,D,\gamma_2\circ\gamma_1)^\circ"] \arrow[d, two heads, "\gamma_1|_{\widehat{X}_1^\circ}"'] & \operatorname{Alb}(X, D, \gamma_2 \circ \gamma_1)^\circ \arrow[d, two heads, "\exists! c^\circ"] \\
\widehat{X}_2^\circ \arrow[r, "\operatorname{alb}(X,D,\gamma_2)^\circ"'] \arrow[d, two heads, "\gamma_2|_{\widehat{X}_2^\circ}"'] & \operatorname{Alb}(X, D, \gamma_2)^\circ \\
X^\circ.
\end{tikzcd} \tag{5.8.1}$$

The morphism c° is quasi-algebraic.

Proof. Uniqueness and surjectivity of c° (if it exists) follows from Proposition 5.5, which asserts that the images of $\operatorname{alb}_\bullet(\bullet)^\circ$ generate $\operatorname{Alb}_\bullet(\bullet)^\circ$ as groups once suitable structures of Lie groups are chosen.

Existence of c° as a quasi-algebraic morphism follows from the universal property of the Albanese. To be precise, recall from Property (5.2.1) that the pull-back morphism

$$\operatorname{alb}(X, D, \gamma_2)^* : H^0\big(\Omega^1_{\operatorname{Alb}(X,D,\gamma_2)}(\log \Delta)\big) \to H^0\Big(\widehat{X}_2,\ \Omega^{[1]}_{\widehat{X}_2}(\log \widehat{D}_1)\Big)$$

takes its image in $H^0\big(\widehat{X}_2,\ \Omega^{[1]}_{(X,D,\gamma_2)}\big)$. As a consequence, we find that the pull-back morphism

$$\big(\operatorname{alb}(X, D, \gamma_2) \circ \gamma_1\big)^* : H^0\big(\Omega^1_{\operatorname{Alb}(X,D,\gamma_2)}(\log \Delta)\big) \to H^0\Big(\widehat{X}_1,\ \Omega^{[1]}_{\widehat{X}_1}(\log \widehat{D}_1)\Big)$$

takes its image in

$$H^0\big(\widehat{X}_1,\ \gamma_1^{[*]} \Omega^{[1]}_{(X,D,\gamma_2)}\big) \subseteq H^0\big(\widehat{X}_1,\ \Omega^{[1]}_{(X,D,\gamma_2\circ\gamma_1)}\big), \tag{5.8.2}$$

where the inclusion in (5.8.2) is [KR24a, Obs. 4.14]. As pointed out above, the universal property of the Albanese $\operatorname{Alb}(X, D, \gamma_2 \circ \gamma_1)^\circ$ now gives a unique quasi-algebraic morphism c° that makes Diagram (5.8.1) commute. □

5.5. **The Albanese of a Galois cover.** Lemma 5.8 applies in particular in case where $\widehat{X}_1 = \widehat{X}_2$ are equal and where γ_1 is a Galois automorphism of the cover γ_2. We find that the Galois group acts on the Albanese and that the Albanese morphism is equivariant.

Observation 5.9 (Galois action on the Albanese of a cover). In Setting 5.1, assume that X is Kähler and that the cover γ is Galois with group G. Recall from [KR24a, Obs. 4.19] that $\Omega^{[1]}_{(X,D,\gamma)}$ carries a natural G-linearisation that is compatible with the natural $\operatorname{Aut}(\widehat{X})$-linearisations of $\Omega^{[1]}_{\widehat{X}}$. In complete analogy to Proposition 4.11, it follows from Lemma 5.8 that G acts on $\operatorname{Alb}(X, D, \gamma)^\circ$ by quasi-algebraic automorphisms, in a way that makes the morphism $\operatorname{alb}(X, D, \gamma)^\circ$ equivariant. Corollary 3.21 on page 8 allows choosing a compactification

$$\operatorname{Alb}(X, D, \gamma)^\circ \subset \operatorname{Alb}(X, D, \gamma), \quad \text{written in short as } A^\circ \subset A,$$

such that the G-action on A° extends to A, and such that $A^\circ \subset A$ is an Albanese for (X, D, γ). □

Construction 5.10 (Morphism to Galois quotient of the Albanese of a cover). In Observation 5.9, take quotients to find a diagram

(5.10.1)
$$\begin{tikzcd}
\widehat{X}^\circ \arrow[r, "\operatorname{alb}_{\widehat{X}}(X,D,\gamma)^\circ"] \arrow[d, two heads, "{\gamma,\ \text{quotient}}"'] & A^\circ \arrow[d, two heads, "{\gamma_A,\ \text{quotient}}"] \\
X^\circ \arrow[r, "a^\circ"'] & A^\circ \big/ G
\end{tikzcd}$$

where a° is quasi-algebraic for the compactifications $X^\circ \subset X$ and $A^\circ/G \subset A/G$. Propositions 5.5 and 3.29 together guarantee that the image of $\operatorname{alb}_{\widehat{X}}(X, D, \gamma)^\circ$ is not contained in the ramification locus of the quotient morphism $\gamma_A : A^\circ \twoheadrightarrow A^\circ/G$. The image of a° is therefore not contained in the branch locus.

Diagram (5.10.1) is a commutative diagram of holomorphic morphisms between normal analytic varieties. We upgrade it to a commutative diagram of $\mathcal{C}$-morphisms.

Observation 5.11 ($\mathcal{C}$-Morphism to Galois quotient of the Albanese). The variety A° of Observation 5.9 and Construction 5.10 is a semitorus and therefore smooth. The criterion for $\mathcal{C}$-morphisms spelled out in [KR24a, Prop. 8.6] therefore applies to show that $\operatorname{alb}_{\widehat{X}}(X, D, \gamma)^\circ$ induces a morphism of $\mathcal{C}$-pairs[3],

$$\operatorname{alb}(X, D, \gamma)^\circ : (\widehat{X}^\circ, 0) \to (A^\circ, 0).$$

Taking the categorical quotients of $\mathcal{C}$-pairs, [KR24a, Prop. 12.7] will thus yield a diagram of $\mathcal{C}$-morphisms between $\mathcal{C}$-pairs as follows,

(5.11.1)
$$\begin{tikzcd}
(\widehat{X}^\circ, 0) \arrow[r, "\operatorname{alb}(X,D,\gamma)"] \arrow[d, two heads, "{\gamma,\ \text{quotient}}"'] & (A^\circ, 0) \arrow[d, two heads, "{\gamma_A,\ \text{quotient}}"] \\
(X^\circ, D') \arrow[r, "a^\circ"'] & (Y^\circ, D_Y),
\end{tikzcd}$$

where

$$(X^\circ, D') := (\widehat{X}^\circ, 0) \big/ G \quad \text{and} \quad (Y^\circ, D_Y) := (A^\circ, 0) \big/ G.$$

Warning 5.12 (Boundary divisors in the quotient construction). The boundary divisor D' in Observation 5.11 does not need to equal D°. In fact, recall from [KR24a, Obs. 12.9] that there is only an inequality $D' \geq D^\circ$, which might be strict. As before, [KR24a, Prop. 10.4] allows formulating this inequality by saying that the identity on X° induces a morphism of $\mathcal{C}$-pairs,

$$\operatorname{Id}_{X^\circ} : (X^\circ, D') \to (X^\circ, D^\circ).$$

Warning 5.12 ends here.

The following proposition, which is central to everything that follows, claims that in spite of Warning 5.12, the morphism a° of Diagram (5.11.1) does induce a morphism of $\mathcal{C}$-pairs,

$$\underline{a}^\circ : (X^\circ, D^\circ) \to (Y^\circ, D_Y).$$

This is expressed in technical terms by saying that the quasi-algebraic $\mathcal{C}$-morphism a° of Diagram (5.11.1) factorizes via the $\mathcal{C}$-morphism $\operatorname{Id}_{X^\circ}$ that we discussed in Warning 5.12.

---

[3] In contrast, recall from [KR24a, Ex. 8.7 and 8.8] that a morphism between singular spaces $Z_1 \to Z_2$ does not always induce a $\mathcal{C}$-morphism $(Z_1, 0) \to (Z_2, 0)$.

Proposition 5.13. *In the setting of Observation 5.9 and Warning 5.12, the quasi-algebraic C-morphism a° of Diagram (5.11.1) factorizes via $\mathrm{Id}_{X^\circ} : (X^\circ, D') \to (X^\circ, D^\circ)$. In other words, we obtain a diagram of C-morphisms,*

$$\begin{tikzcd}
(\widehat{X}^\circ, 0) \arrow[rr, "\mathrm{alb}(X,D,\gamma)^\circ"] \arrow[d, two heads, "\gamma,\ \textit{quotient}"'] & & (A^\circ, 0) \arrow[d, two heads, "\gamma_A,\ \textit{quotient}"] \\
(X^\circ, D') \arrow[r, "\mathrm{Id}_{X^\circ}"] \arrow[rr, bend right, "a^\circ"'] & (X^\circ, D^\circ) \arrow[r, "\underline{a}^\circ"] & (Y^\circ, D_Y),
\end{tikzcd}$$

where $\operatorname{img} a^\circ = \operatorname{img} \underline{a}^\circ$ *is not contained in the branch locus of the quotient morphism* γ_A.

Proof. We aim to apply the criterion for C-morphisms spelled out in [KR24a, Prop. 9.3] and consider the sub-diagram

$$\begin{tikzcd}
\widehat{X}^\circ \arrow[r, "\mathrm{alb}(X,D,\gamma)^\circ"] \arrow[d, two heads, "\gamma,\ \text{quotient}"'] & A^\circ \arrow[d, two heads, "\gamma_A,\ \text{quotient}"] \\
X^\circ \arrow[r, "\underline{a}^\circ"'] & Y^\circ.
\end{tikzcd}$$

Recall [KR24a, Obs. 12.10], which asserts that γ_A is strongly adapted for the C-pair (Y°, D_Y), and that the C-cotangent sheaf is $\Omega^{[1]}_{(Y^\circ, D_Y, \gamma_A)} = \Omega^1_{A^\circ}$. Given that A° is a semi-torus, we find that $\Omega^{[1]}_{(Y^\circ, D_Y, \gamma_A)}$ is locally free. The criterion for C-morphisms, [KR24a, Prop. 9.3] therefore applies to show that $\underline{a}^\circ$ is a C-morphism as soon as we show that there exists a sheaf morphism

$$d\,\mathrm{alb}(X, D, \gamma)^\circ : \left(\mathrm{alb}(X, D, \gamma)^\circ\right)^* \Omega^{[1]}_{(Y^\circ, D_Y, \gamma_A)} \to \Omega^{[1]}_{(X^\circ, D^\circ, \gamma)}$$

that agrees with the standard pull-back of Kähler differentials wherever this makes sense. That is however precisely the statement of Remark 5.3. The fact that $\operatorname{img} a^\circ = \operatorname{img} \underline{a}^\circ$ is not contained in the branch locus of the quotient morphism γ_A has already been remarked in Construction 5.10. □

5.6. **Functoriality in sequences of Galois covers.** The following lemma combines and summarizes the results of Sections 5.4 and 5.5.

Lemma 5.14 (Functoriality of the Albanese). *In the setting of Lemma 5.8, assume that the covering morphisms $\gamma_2 \circ \gamma_1$ and γ_2 are Galois, with groups G_2 and G_1 respectively. Then, there exists a commutative diagram*

$$\begin{tikzcd}
\widehat{X}_1^\circ \arrow[r, "\mathrm{alb}(X,D,\gamma_2\circ\gamma_1)^\circ"] \arrow[d, two heads, "\gamma_1|_{\widehat{X}_1^\circ}"'] & \mathrm{Alb}(X, D, \gamma_2 \circ \gamma_1)^\circ \arrow[dr, two heads, bend left, "c^\circ,\ \textit{quot. of semitori}"] \arrow[dd, two heads, "\textit{quotient}"] & \\
\widehat{X}_2^\circ \arrow[rr, "\mathrm{alb}(X,D,\gamma_2)^\circ"'] \arrow[d, two heads, "\gamma_2|_{\widehat{X}_2^\circ}"'] & & \mathrm{Alb}(X, D, \gamma_2)^\circ \arrow[d, two heads, "\textit{quotient}"] \\
X^\circ \arrow[r, "\underline{\mathrm{alb}}(X,D,\gamma_2\circ\gamma_1)^\circ"] \arrow[rr, bend right, "\underline{\mathrm{alb}}(X,D,\gamma_2)^\circ"'] & \mathrm{Alb}(X, D, \gamma_2 \circ \gamma_1)^\circ \big/ G_1 \arrow[r, two heads, "\underline{c}^\circ"] & \mathrm{Alb}(X, D, \gamma_2)^\circ \big/ G_2
\end{tikzcd}$$

where all morphisms are quasi-algebraic and all morphisms in the bottom row are morphisms of C-pairs, between (X°, D°) and the natural C-structures on the quotient pairs.

Proof. Except for the morphism $\underline{c}^\circ$, the diagram is a combination of Lemma 5.8 and Proposition 5.13 above. In order to construct $\underline{c}^\circ$, observe that the group G_2 is a quotient $q : G_1 \twoheadrightarrow G_2$, and that G_1 acts on $\widehat{X}_2^\circ$ via this quotient map, in a manner that makes the morphism $\gamma_1|_{\widehat{X}_1^\circ}$ equivariant. The universal properties of the two Albanese maps $\operatorname{alb}(X, D, \gamma_2 \circ \gamma_1)^\circ$ and $\operatorname{alb}(X, D, \gamma_2)^\circ$ will then guarantee that G_1 acts on the Albanese varieties $\operatorname{Alb}(X, D, \gamma_2 \circ \gamma_1)^\circ$ and $\operatorname{Alb}(X, D, \gamma_2)^\circ$ in a manner that makes the quotient morphism c° equivariant. The map $\underline{c}^\circ$ is then the induced $\mathcal{C}$-morphism between the quotients pairs, as given by the universal property of $\mathcal{C}$-pair quotients, [KR24a, Def. 12.3 and Thm. 12.5]. □

6. The Albanese for a subspace of differentials

This section proves the existence of an Albanese of a cover as a special case of the "Albanese for a subspace of differentials". We refer the reader to [Zuo99, Sect. 4.2] for a related construction in the smooth, proper case. Throughout the present section, we work in following setting.

Setting 6.1. Let (X, D) be a log pair where X is compact. Let $V \subseteq H^0\Big(X,\, \Omega_X^{[1]}(\log D)\Big)$ be a linear subspace.

Definition 6.2 (The Albanese for a subspace of differentials)**.** *Assume Setting 6.1. An Albanese of (X, D, V) is a semitoric variety $\operatorname{Alb}(X, D, V)^\circ \subset \operatorname{Alb}(X, D, V)$ together with a quasi-algebraic morphism*

$$\operatorname{alb}(X, D, V)^\circ : X^\circ \to \operatorname{Alb}(X, D, V)^\circ$$

such that the following holds.

(6.2.1) *The pull-back morphism of logarithmic differentials,*

$$d\operatorname{alb}(X, D, V) : H^0\Big(\Omega^1_{\operatorname{Alb}(X,D,V)}(\log \Delta)\Big) \to H^0\Big(X,\, \Omega_X^{[1]}(\log D)\Big)$$

takes its image in V.

(6.2.2) *If $A^\circ \subset A$ is any semitoric variety and if $a^\circ : X^\circ \to A^\circ$ is quasi-algebraic such that*

$$d\, a : H^0\big(A,\, \Omega^1_A(\log \Delta)\big) \to H^0\Big(X,\, \Omega_X^{[1]}(\log D)\Big)$$

takes its image in V, then a° factors uniquely as

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D,V)^\circ"'] \arrow[rr, bend left, "a^\circ"] & \operatorname{Alb}(X, D, V)^\circ \arrow[r, "\exists! b^\circ"'] & A^\circ, \end{tikzcd}$$

where b° is quasi-algebraic.

Warning 6.3. We do not claim or ask in Item (6.2.1) that the space V is equal to the image of the morphism $d\operatorname{alb}(X, D, V)$. See Section 6.3 on page 24 for a sobering example which shows that surjectivity is a delicate property of the subspace V.

We will later consider Definition 6.2 in a setting where the space V is of the form $V = H^0(X,\, \mathscr{F})$, for a subsheaf $\mathscr{F} \subseteq \Omega^1_X(\log D)$. The following notion will be used.

Definition 6.4 (The Albanese for subsheaves of differentials)**.** *Assume Setting 6.1. If there exists a subsheaf $\mathscr{F} \subseteq \Omega^1_X(\log D)$ such that $V = H^0(X,\, \mathscr{F})$, then we denote the Albanese briefly as* $\operatorname{alb}(X, D, \mathscr{F})^\circ : X^\circ \to \operatorname{Alb}(X, D, \mathscr{F})^\circ$.

6.1. **Uniqueness and existence.** As before, the universal property spelled out in Item (6.2.2) implies that $\operatorname{Alb}(X, D, V)^\circ$ is unique up to unique isomorphism and that $\operatorname{Alb}(X, D, V)$ is bimeromorphically unique. As before, we abuse notation and refer to any Albanese as "the Albanese", with associated semitoric *Albanese variety* $\operatorname{Alb}(X, D, V)^\circ \subset \operatorname{Alb}(X, D, V)$ and quasi-algebraic *Albanese morphism* $\operatorname{alb}(X, D, V)^\circ$.

Proposition 6.5. *Assume Setting 6.1. If X is Kähler, then an Albanese of (X, D, V) exists. The dimension is bounded by*

$$\dim \operatorname{Alb}(X, D, V)^\circ \le \dim_{\mathbb{C}} V. \tag{6.5.1}$$

If $x \in X^\circ$ is any point and if we use

$$0_{\operatorname{Alb}(X,D,V)^\circ} := \operatorname{alb}(X, D, V)^\circ(x) \in \operatorname{Alb}(X, D, V)^\circ$$

to equip $\operatorname{Alb}(X, D, V)^\circ$ with the structure of a Lie group, then the image of $\operatorname{alb}(X, D, V)^\circ$ generates $\operatorname{Alb}(X, D, V)^\circ$ as an Abelian group.

Example 6.8 on page 24 shows that Inequality (6.5.1) might be strict. As in Section 4.2, we give a direct construction of one Albanese.

Construction 6.6 (Construction of the Albanese for a subspace of differentials). In Setting 6.1, consider the annihilator $V^\perp \subseteq H^0\big(X,\, \Omega_X^{[1]}(\log D)\big)^*$ and recall from Observation 4.15 that the construction of $\operatorname{Alb}(X, D)^\circ$ equips us with a canonical holomorphic Lie group morphism

$$H^0\big(X,\, \Omega_X^{[1]}(\log D)\big)^* \twoheadrightarrow \operatorname{Alb}(X, D)^\circ. \tag{6.6.1}$$

The image

$$I_V := \operatorname{img}\big(V^\perp \to \operatorname{Alb}(X, D)^\circ\big)$$

is then a subgroup of $\operatorname{Alb}(X, D)$ that may or may not be closed. Either way, Fact 3.26 on page 9 allows taking the smallest quasi-algebraic subgroup $B \subseteq \operatorname{Alb}(X, D)^\circ$ that contains I_V. We write

$$\operatorname{Alb}(X, D, V)^\circ := \operatorname{Alb}(X, D)^\circ \big/ B$$

and obtain morphisms

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"'] \arrow[rr, bend left, "\operatorname{alb}(X,D,V)^\circ"] & \operatorname{Alb}(X, D)^\circ \arrow[r, two heads, "q^\circ\text{, quotient}"'] & \operatorname{Alb}(X, D, V)^\circ. \end{tikzcd}$$

Recall from Facts 3.23 and 3.27 that B and $\operatorname{Alb}(X, D, V)$ are isomorphic to semitori, and that there exists a semitoric compactification $\operatorname{Alb}(X, D, V)^\circ \subseteq \operatorname{Alb}(X, D, V)$ that renders the quotient morphism q° quasi-algebraic. With this choice of compactification, Lemma 2.4 guarantees that the morphism $\operatorname{alb}(X, D, V)^\circ$ is quasi-algebraic, as desired.

Proof of Proposition 6.5. We need to verify that Construction 6.6 satisfies the properties spelled out in Proposition 5.5. Once this is done, Proposition 4.10 on page 13 and surjectivity of the quotient morphism q guarantees that the image of $\operatorname{alb}(X, D, V)^\circ$ generates $\operatorname{Alb}(X, D, V)^\circ$ as an Abelian group, as claimed.

Property (6.2.1). To prove Property (6.2.1), write $W := \operatorname{img} d \operatorname{alb}(X, D, V)$ and recall that

$$\operatorname{img}\big(V^\perp \to \operatorname{Alb}(X, D)^\circ\big) \overset{\text{constr.}}{\subseteq} \ker(q^\circ) \overset{\text{Fact 4.19}}{=} \operatorname{img}\big(W^\perp \to \operatorname{Alb}(X, D)^\circ\big).$$

Given that the Lie group morphism (6.6.1) has maximal rank, we find that $V^\perp \subseteq W^\perp$ and hence that $V \supseteq W$, as desired.

Property (6.2.2). Assume that a morphism $a^\circ : X^\circ \to A^\circ$ as in Property (6.2.2) is given. The universal property of $\operatorname{Alb}(X,D)$ will then yield a factorization

$$\begin{tikzcd} X^\circ \arrow[r, "\operatorname{alb}(X,D)^\circ"'] \arrow[rr, bend left, "a^\circ"] & \operatorname{Alb}(X,D)^\circ \arrow[r, "\beta^\circ\text{, quasi-algebraic}"'] & A^\circ. \end{tikzcd}$$

We claim that the quasi-algebraic morphism β° factors via q°,

$$\begin{tikzcd} \operatorname{Alb}(X,D)^\circ \arrow[r, two heads, "q^\circ\text{, quasi-algebraic}"'] \arrow[rr, bend left, "\beta^\circ\text{, quasi-algebraic}"] & \operatorname{Alb}(X,D)^\circ \big/ B \arrow[r, "\exists ! b^\circ"'] & A^\circ. \end{tikzcd}$$

Equivalently, we claim that $B \subseteq \ker(\beta^\circ)$. This follows easily: writing

$$W := \operatorname{img}\Bigl(d\, a : H^0\bigl(A,\, \Omega^1_A(\log \Delta)\bigr) \to H^0\bigl(X,\, \Omega^{[1]}_X(\log D)\bigr)\Bigr),$$

we know by assumption that $W \subseteq V$ or equivalently, that $W^\perp \supseteq V^\perp$. By Fact 4.19 on page 15, this is in turn equivalent to $\ker(\beta) \supseteq I_V$. The desired inclusion $\ker(\beta^\circ) \supseteq B$ follows as soon as we recall from Fact 3.28 on page 9 that $\ker(\beta^\circ)$ is quasi-algebraic. Lemma 2.4 on page 3 guarantees that b° is quasi-algebraic, as required. The statement about the dimension is clear from the construction. □

6.2. **Proof of Proposition 5.5.** In the setting of Proposition 5.5, set

$$V := H^0\bigl(\widehat{X},\, \Omega^{[1]}_{(X,D,\gamma)}\bigr).$$

Using the notation introduced in Definition 6.4, Proposition 6.5 equips us with a semitoric variety

$$\operatorname{Alb}\left(\widehat{X}, \widehat{D}, \Omega^{[1]}_{(X,D,\gamma)}\right)^\circ \subset \operatorname{Alb}\left(\widehat{X}, \widehat{D}, \Omega^{[1]}_{(X,D,\gamma)}\right)$$

and a quasi-algebraic morphism

$$\operatorname{alb}\left(\widehat{X}, \widehat{D}, \Omega^{[1]}_{(X,D,\gamma)}\right)^\circ : \widehat{X}^\circ \to \operatorname{Alb}\left(\widehat{X}, \widehat{D}, \Omega^{[1]}_{(X,D,\gamma)}\right)^\circ$$

that we take as the Albanese of the cover γ of the $\mathcal{C}$-pair (X,D). A comparison of the Properties (6.2.1)–(6.2.2) guaranteed by Proposition 6.5 with the Properties (5.2.1)–(5.2.2) required by Proposition 5.5 concludes the proof. □

6.3. **Examples.** We end the present section with two simple examples.

Example 6.7. In the setting of Definition 6.2, if $V = \{0\}$, then $\operatorname{Alb}(X,D,V)$ is a point.

Example 6.8. Let E be an elliptic curve. Set $X = E \times E$ and take $D := 0 \in \operatorname{Div}(X)$. Pulling back differentials from the two factors gives natural morphisms

$$d\, \pi_i : H^0\bigl(E,\, \Omega^1_E\bigr) \to H^0\bigl(X,\, \Omega^1_X\bigr).$$

Choose a number $\tau \in \mathbb{C}$ and set $V := \operatorname{img}\bigl((d\, \pi_1) + \tau \cdot (d\, \pi_2)\bigr)$, which is a one-dimensional linear subspace of $H^0\bigl(X,\, \Omega^1_X\bigr)$. The following will hold.

- If τ is non-rational, then I_V is dense in $\operatorname{Alb}(X,0)^\circ$ and $\operatorname{Alb}(X,0,V)^\circ = \{0\}$.
- Towards the other extreme, if $\tau = 0$, then $\operatorname{Alb}(X,0,V)^\circ = \operatorname{Alb}(E)$.

7. Boundedness of the Albanese irregularity for special pairs

Following Ueno's work [Uen75], Campana has remarked in [Cam04, Sect. 5.2] that the Albanese morphism of a special manifold is always surjective. We extend Campana's observation to the Albanese of a cover. For $\mathcal{C}$-pairs that are special in the sense of [KR24a, Def. 6.11], the following theorem implies that the dimension of the Albanese is bounded by the dimension of X. In particular, it cannot go to infinity as we consider higher and higher covers. Along these lines, we view the theorem as a boundedness result.

Theorem 7.1 ($\mathcal{C}$-pairs whose Albanese morphism is not dominant)**.** *In Setting 5.1, assume that X is Kähler. If the Albanese morphism $\operatorname{alb}(X, D, \gamma)^\circ$ is* not *dominant, then there exists a number $1 \le p \le \dim X$ and a coherent rank-one subsheaf $\mathscr{L}_1 \subset \Omega^{[p]}_{(X,D,\mathrm{Id}_X)}$ with $\mathcal{C}$-Kodaira-Iitaka dimension $\kappa_{\mathcal{C}}(\mathscr{L}_1) \ge p$.*

We refer the reader to [KR24a, Sect. 6] for the definition of "$\mathcal{C}$-Kodaira-Iitaka dimension" and for the related notions of "special pairs" and "Bogomolov sheaves".

Corollary 7.2 (The Albanese for covers for special pairs)**.** *In Setting 5.1, assume that X is Kähler. If (X, D) is special, then the Albanese morphism $\operatorname{alb}(X, D, \gamma)^\circ$ is dominant.* □

The proof of Theorem 7.1 is given in Section 7.2, starting from Page 27 below.

Remark 7.3. Recall from Definition 5.2 that the Albanese morphism $\operatorname{alb}(X, D, \gamma)^\circ$ is quasi-algebraic, so that topological closure of its image,

$$\overline{\operatorname{img}\operatorname{alb}(X, D, \gamma)^\circ} \subseteq \operatorname{Alb}(X, D, \gamma)^\circ,$$

is always analytic. The word "dominant" in Theorem 7.1 and Corollary 7.2 is therefore meaningful.

Remark 7.4. Assume Setting 5.1. If the $\mathcal{C}$-pair (X, D) is special, Corollary 7.2 implies in particular that $q^+_{\mathrm{Alb}}(X, D, \gamma) \le \dim X$.

Even for special pairs, one cannot expect that the Albanese morphism $\operatorname{alb}(X, D, \gamma)^\circ$ is surjective. The following simple example shows what can go wrong.

Example 7.5 (Failure of surjectivity). Let T be a compact torus and let $t \in T$ be any point. Let X be the blow-up of T in t and let $D \in \operatorname{Div}(X)$ be the exceptional divisor, with multiplicity one. Then, the logarithmic pair (X, D) is special, the Albanese for the identity morphism equals $\operatorname{Alb}(X, D, \mathrm{Id}_X)^\circ = T$ and

$$\operatorname{img}\operatorname{alb}(X, D, \gamma)^\circ = T \setminus \{t\}.$$

7.1. **Failure of dominance.**

To prepare for the proof of Theorem 7.1, we analyse the setting where the Albanese of a cover fails to be dominant. The construction presented here will also be used in the forthcoming paper [KR24b], where we prove a $\mathcal{C}$-version of the Bloch-Ochiai theorem.

Setting 7.6 (Failure of dominance). In Setting 5.1, assume that X is Kähler. Assume that the cover γ is Galois with group G, and use Corollary 3.21 on page 8 to choose an Albanese

$$\operatorname{Alb}(X, D, \gamma)^\circ \subset \operatorname{Alb}(X, D, \gamma), \quad \text{written in short as } \operatorname{Alb}^\circ \subset \operatorname{Alb},$$

such that the G-action on Alb° extends to Alb. Recall that the Albanese morphism alb° is quasi-algebraic. The topological closure of the image, $\aleph := \overline{\operatorname{img}\operatorname{alb}^\circ}$, is thus an analytic subset of Alb. Set $\aleph^\circ := \aleph \cap \operatorname{Alb}^\circ$ and assume that $\aleph^\circ$ is a proper subset, $\aleph^\circ \subsetneq \operatorname{Alb}^\circ$. Finally, choose an element $\widehat{x} \in \widehat{X}^\circ$ and use its image point

$$0_{\operatorname{Alb}^\circ} := \operatorname{alb}^\circ(\widehat{x}) \in \operatorname{Alb}^\circ$$

to equip Alb° with the structure of a Lie group.

Remark 7.7 (Stabilizer groups). In Setting 7.6, recall from [NW14, Prop. 5.3.16] that the stabilizer subgroup

$$\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ) = \left\{ a \in \mathrm{Alb}^\circ \mid a + \aleph^\circ = \aleph^\circ \right\} \subset \mathrm{Alb}^\circ$$

is closed and quasi-algebraic. Recall from [NW14, Prop. 5.3.13] that its maximal connected subgroup $I \subset \mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)$ is then a semitorus.

Observation 7.8 (Properness of I as a subgroup of Alb°). By construction, we have

$$0_{\mathrm{Alb}^\circ} = \mathrm{alb}^\circ(\widehat{x}) \in \operatorname{img} \mathrm{alb}^\circ \subseteq \aleph^\circ.$$

It follows that $\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ) \subseteq \aleph^\circ$. This equips us with inclusions

$$I \subseteq \mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ) \subseteq \aleph^\circ \subsetneq \mathrm{Alb}^\circ$$

and shows that $I \subsetneq \mathrm{Alb}^\circ$ is a proper subgroup. The quotient group Alb° / I is not trivial.

We have seen in Observation 5.9 on page 19 that the morphism alb° is equivariant with respect to the G-action on Alb°. The action will then stabilize the subset $\aleph^\circ$. As the next lemma shows, it will also stabilize $\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)$ and I, at least up to translation.

Lemma 7.9 (Relation between G and I)**.** *In Setting 7.6, if $g \in G$ is any element, then $g \cdot I$ is a translate of I. In particular, the G-action of* Alb° *maps I-orbits to I-orbits, for the additive action of I on* Alb°.

Proof. Since all connected components of the group $\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)$ are translates of the identity component, it suffices to show that $g \cdot \mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)$ is a translate of $\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)$. To this end, recall from Proposition 3.18 on page 7 that we may write $g : \mathrm{Alb}^\circ \to \mathrm{Alb}^\circ$ in the form $g : a \mapsto f^\circ(a) + g(0_{\mathrm{Alb}^\circ})$, where $f^\circ : \mathrm{Alb}^\circ \to \mathrm{Alb}^\circ$ is a group morphism. In particular, we find that

$$\aleph^\circ = g(\aleph^\circ) = f^\circ(\aleph^\circ) + g(0_{\mathrm{Alb}^\circ}) \quad \Leftrightarrow \quad f^\circ(\aleph^\circ) = \aleph^\circ - g(0_{\mathrm{Alb}^\circ}). \tag{7.9.1}$$

This gives

$$\begin{aligned} g\big(\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)\big) &= f^\circ\big(\mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ)\big) + g(0_{\mathrm{Alb}^\circ}) \\ &= \mathrm{St}_{\mathrm{Alb}^\circ}\big(f^\circ(\aleph^\circ)\big) + g(0_{\mathrm{Alb}^\circ}) && f^\circ \text{ a group morphism} \\ &= \mathrm{St}_{\mathrm{Alb}^\circ}\big(\aleph^\circ - g(0_{\mathrm{Alb}^\circ})\big) + g(0_{\mathrm{Alb}^\circ}) && (7.9.1) \\ &= \mathrm{St}_{\mathrm{Alb}^\circ}(\aleph^\circ) + g(0_{\mathrm{Alb}^\circ}) && \text{Defn. of } \mathrm{St}_{\mathrm{Alb}^\circ}(\bullet) \end{aligned}$$

□

Construction 7.10. Maintaining Setting 7.6, we construct a non-trivial semitoric variety $B^\circ \subset B$ with G-action and a diagram

$$\begin{tikzcd} \widehat{X}^\circ \arrow[r, "\mathrm{alb}^\circ"'] \arrow[rr, bend left, two heads, "b^\circ"] \arrow[d, two heads, "\gamma^\circ\text{, quotient by } G"'] & \mathrm{Alb}^\circ \arrow[r, two heads, "\beta^\circ\text{, quotient by } I"'] \arrow[d, two heads, "\gamma_{\mathrm{Alb}^\circ}\text{, quotient by } G"] & B^\circ \arrow[d, two heads, "\gamma_{B^\circ}\text{, quotient by } G"] \\ X^\circ \arrow[r, "\delta^\circ"'] & \mathrm{Alb}^\circ \big/ G \arrow[r, two heads, "\varepsilon^\circ"'] & B^\circ \big/ G \end{tikzcd}$$

where (among other things) the following holds.

- All horizontal arrows are quasi-algebraic,
- all arrows in the top row are G-equivariant, and
- all arrows in the bottom row are C-morphisms for the C-pairs

$$(X^\circ, D^\circ), \quad \left(\mathrm{Alb}^\circ, 0\right) \big/ G, \quad \text{and} \quad \left(B^\circ, 0\right) \big/ G.$$

The left rectangle of the diagram is given by Proposition 5.13 on page 20. As for the right rectangle, take B° as the quotient Alb°/I. Recall from [NW14, Thm. 5.3.13] that B° is a semitorus, and that there exists a semitoric compactification $B^\circ \subseteq B$ that renders the quotient morphism β° quasi-algebraic. Lemma 7.9 gives a natural action $G \circlearrowleft B^\circ$ that makes the morphism β° equivariant, and Corollary 3.21 on page 8 allows assuming without loss of generality that the G action extends from B° to B. The right rectangle of the diagram is now given by the universal property of G-quotients, [KR24a, Prop. 12.7]. Finish the construction by recalling from [KR24a, Prop. 12.7] that ε° is a morphism of C-pairs, from $\big(\mathrm{Alb}^\circ, 0\big)/G$ to $\big(B^\circ, 0\big)/G$, as required.

To conclude Construction 7.10, consider the topological closure $Z := \overline{\operatorname{img}\beta^\circ}$, which is an analytic subset of B. As before, write $Z^\circ := Z \cap B^\circ$ and set $p := \dim Z$.

The following observations summarize the main properties of the construction.

Observation 7.11. By construction, Z° is not invariant under the action of any proper semitorus in B°. In this setting, recall from Kawamata's proof of the Bloch conjecture, [Kaw80], or more specifically from [Kob98, Cor. 3.8.27] that there exist B°-invariant differentials $\tau_0^\circ, \dots, \tau_p^\circ \in H^0\big(B^\circ,\, \Omega^p_{B^\circ}\big)$ such that the restrictions $\tau_\bullet^\circ|_{Z^\circ_{\mathrm{reg}}}$ are linearly independent top-differentials on Z°_{reg}, and therefore define a $(p+1)$-dimensional linear subspace

$$V := \big\langle \tau_0^\circ|_{Z^\circ_{\mathrm{reg}}}, \dots, \tau_p^\circ|_{Z^\circ_{\mathrm{reg}}} \big\rangle \subseteq H^0\big(Z^\circ_{\mathrm{reg}},\, \Omega^p_{Z^\circ_{\mathrm{reg}}}\big) = H^0\big(Z^\circ_{\mathrm{reg}},\, \omega_{Z^\circ_{\mathrm{reg}}}\big).$$

The associated meromorphic map $\varphi_V : Z^\circ_{\mathrm{reg}} \dashrightarrow \mathbb{P}^p$ is generically finite. Recall from Item (3.15.2) of Proposition 3.15 on page 6 that the B°-invariant differentials $\tau_\bullet^\circ \in H^0\big(B^\circ,\, \Omega^p_{B^\circ}\big)$ automatically extend to differentials with logarithmic poles at infinity, say $\tau_\bullet \in H^0\big(B,\, \Omega^p_B(\log \Delta)\big)$.

Observation 7.12. We have observed in 7.8 that $I \subseteq \aleph^\circ$. There is more that we can say. The assumption $\aleph^\circ \subsetneq \mathrm{Alb}^\circ$ and Item (5.2.2) of Definition 5.2 imply that $\aleph^\circ$ is not itself a semitorus. In particular, we find that $I \subsetneq \aleph^\circ$ is a proper subset and that the variety Z° is therefore positive-dimensional. The inclusion $I \subset \aleph^\circ$ also implies that the morphisms

$$\beta^\circ : \mathrm{Alb}^\circ \twoheadrightarrow B^\circ \quad \text{and} \quad \beta^\circ|_{\aleph^\circ} : \aleph^\circ \to Z^\circ$$

are G-equivariant fibre bundles, both with typical fibre I. The analytic variety Z° is therefore a proper subset, $Z^\circ \subsetneq B^\circ$.

7.2. **Proof of Theorem 7.1.** We prove Theorem 7.1 in the remainder of the present Section 7 and maintain Setting 5.1 throughout. For simplicity of notation, we prove the contrapositive: assuming that the Albanese morphism $\mathrm{alb}(X, D, \gamma)^\circ$ is *not* dominant, we show that the C-pair (X, D) admits a Bogomolov sheaf and is hence *not* special.

The proof follows classic arguments, with some additional complications because of our use of adapted differentials and because of the singularities of the varieties involved.

Step 1: Simplification. Recall Lemma 5.8: Non-dominance of $\mathrm{alb}(X, D, \gamma)^\circ$ is preserved when we replace γ by any cover that factors via γ. We can therefore pass to the Galois closure and assume that we are in Setting 7.6. We use the notation introduced in Construction 7.10 and Observations 7.11–7.12 in the remainder of the proof.

Step 2: A rank-one sheaf in $\Omega^{[p]}_{(X,D,\gamma)}|_{\widehat{X}^\circ}$. Consider the composed morphism of G-sheaves

$$(b^\circ)^*\, \Omega^p_{B^\circ} \xrightarrow{\mathrm{d}\, b^\circ} \Omega^p_{\widehat{X}^\circ} \longrightarrow \Omega^{[p]}_{\widehat{X}^\circ}, \tag{7.13.1}$$

and let $\mathscr{L}^\circ \subseteq \Omega^{[p]}_{\widehat{X}^\circ}$ denote the image sheaf, which is then a torsion free G-subsheaf of $\Omega^{[p]}_{\widehat{X}}$. We summarize its main properties.

Observation 7.14. The sheaf $\mathscr{L}^\circ$ is of rank one because b° factors via the p-dimensional space Z°. □ (Observation 7.14)

Claim 7.15. The sheaf $\mathscr{L}^\circ$ is contained in the subsheaf $\Omega^{[p]}_{(X,D,\gamma)}\big|_{\widehat{X}^\circ} \subseteq \Omega^{[p]}_{\widehat{X}^\circ}$.

Proof of Claim 7.15. The morphism b° factors via alb°. Since pull-back of Kähler differentials is functorial, $d\,b^\circ$ factors via $d\operatorname{alb}^\circ$ and the image of the composed morphism (7.13.1) is contained in the image of the composition

$$(\operatorname{alb}^\circ)^*\,\Omega^p_{\operatorname{Alb}^\circ} \xrightarrow{d\operatorname{alb}^\circ} \Omega^p_{\widehat{X}^\circ} \longrightarrow \Omega^{[p]}_{\widehat{X}^\circ}.$$

But then Remark 5.3 gives the claim. □ (Claim 7.15)

Step 3: A rank-one sheaf in $\Omega^{[p]}_{(X,D,\gamma)}$. We extend the sheaf $\mathscr{L}^\circ$ from $\widehat{X}^\circ$ to a rank-one, reflexive sheaf that is defined on all of $\widehat{X}$. As in Section 4, the reader coming from algebraic geometry might find the proof surprisingly complicated: in the analytic setting, it is typically not possible to extend coherent sheaves across codimension-two subsets.

Claim 7.16. There exists a rank-one, reflexive G-subsheaf $\mathscr{L} \subseteq \Omega^{[p]}_{(X,D,\gamma)}$ whose restriction to $\widehat{X}^\circ$ contains $\mathscr{L}^\circ$. There are sections $\sigma_0, \dots, \sigma_p \in H^0\big(\widehat{X},\, \mathscr{L}\big)$ whose associated linear system defines a dominant meromorphic map $\widehat{X} \dashrightarrow \mathbb{P}^p$.

Proof of Claim 7.16. The morphism $b^\circ : \widehat{X}^\circ \to B^\circ$ is quasi-algebraic and therefore extends to a G-equivariant meromorphic map $b : \widehat{X} \dashrightarrow B$. Choose a G-equivariant log-resolution $(\widetilde{X}, \widetilde{D})$ of $(\widehat{X}, \widehat{D})$ and the meromorphic map b as follows:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, two heads, "{\rho,\ \text{resolution}}"'] \arrow[dr, bend left, "\widetilde{b}"] & \\
\widehat{X} \arrow[r, dashed, "b"'] & B.
\end{tikzcd}$$

We can then consider G-subsheaves

$$\operatorname{img}\Big(d\,\widetilde{b} : \Omega^p_B(\log \Delta) \to \Omega^p_{\widetilde{X}}(\log \widetilde{D})\Big) \subseteq \Omega^p_{\widetilde{X}}(\log \widetilde{D})$$

and

$$\mathscr{L}' := \rho_* \operatorname{img}\big(d\,\widetilde{b}\big) \subseteq \rho_*\Omega^p_{\widetilde{X}}(\log \widetilde{D}) \subseteq \Omega^{[p]}_{\widehat{X}}(\log \widehat{D})$$

The construction guarantees that the sheaves $\mathscr{L}'$ and $\mathscr{L}^\circ$ agree over the open set $\widehat{X}^\circ$; in particular, we find that $\mathscr{L}'$ is of rank one. Together with Claim 7.15, the construction shows that $\mathscr{L}'$ is contained in $\Omega^{[p]}_{(X,D,\gamma)}$. Finally, let $\mathscr{L}$ be the saturation of $\mathscr{L}'$ in $\Omega^{[p]}_{(X,D,\gamma)}$. The sheaf $\mathscr{L}$ is then automatically reflexive. In summary, we obtain inclusions of G-sheaves as follows,

$$\mathscr{L}' \subseteq \mathscr{L} \subseteq \Omega^{[p]}_{(X,D,\gamma)} \subseteq \Omega^{[p]}_{\widehat{X}}(\log \widehat{D}).$$

In order to construct the sections $\sigma_\bullet$, recall from Observation 7.11 that the differentials $\tau_\bullet$ have logarithmic poles at infinity, and then so do their pull-backs. To be more precise, consider the reflexive differentials

$$\sigma_\bullet \in H^0\big(\widehat{X},\, \Omega^{[p]}_{\widehat{X}}(\log \widehat{D})\big)$$

that generically agree with the pull-back of $\tau_\bullet$, and therefore restrict to sections

$$\sigma_\bullet|_{\widehat{X}^\circ} \in H^0\big(\widehat{X}^\circ,\, \mathscr{L}^\circ\big) \subset H^0\big(\widehat{X}^\circ,\, \Omega^{[p]}_{(X,D,\gamma)}\big).$$

But that already implies that the $\sigma_\bullet$ are sections of $\mathscr{L}$. □ (Claim 7.16)

Step 4: Conclusion. Given any number $i \in \mathbb{N}$, recall from [KR24a, Obs. 4.12] that reflexive symmetric multiplication of adapted reflexive tensors yields inclusions

$$\mathscr{L}^{[\otimes i]} \subseteq \operatorname{Sym}^{[i]}_C \Omega^{[p]}_{(X,D,\gamma)}.$$

We consider the G-invariant push-forward sheaves,

$$\mathscr{L}_i := \left(\gamma_* \mathscr{L}^{[\otimes i]}\right)^G \subseteq \left(\gamma_* \operatorname{Sym}^{[i]}_C \Omega^{[p]}_{(X,D,\gamma)}\right)^G \overset{\text{[KR24a, Cor. 4.20]}}{=} \operatorname{Sym}^{[i]}_C \Omega^{[p]}_{(X,D,\mathrm{Id}_X)}.$$

Recall from [GKKP11, Lem. A.4] that the sheaves $\mathscr{L}_i$ are reflexive. By construction, their rank is one. We will show in this step that $\kappa_C(\mathscr{L}_1) \geq p$.

Observation 7.17. If $i \in \mathbb{N}$ is any number, then $\mathscr{L}_i$ equals the i^{th} C-product sheaf

$$\mathscr{L}_i = \operatorname{Sym}^{[i]}_C \mathscr{L}_1,$$

as introduced in [KR24a, Def. 6.5] □

Recalling the definition of the C-Kodaira-Iitaka dimension from [KR24a, Sect. 6.2], it remains to find one sheaf $\mathscr{L}_i$ with non-empty linear system whose associated meromorphic map has an image of dimension $\geq p$. For this, consider the linear systems

$$W_i := H^0\left(\widehat{X}, \mathscr{L}^{[\otimes i]}\right)^G \subseteq H^0\left(\widehat{X}, \mathscr{L}^{[\otimes i]}\right).$$

If i is sufficiently large and divisible, then W_i is positive-dimensional and the associated meromorphic map $\varphi_W : \widehat{X} \dashrightarrow \mathbb{P}^\bullet$ has an image of dimension

$$\dim \operatorname{img} \varphi_W \geq \dim \operatorname{img}(\varphi_V \circ b^\circ) \geq p.$$

By construction, the meromorphic map φ_W is constant on G-orbits and the induced meromorphic map $\varphi : X \dashrightarrow \mathbb{P}^\bullet$ equals the meromorphic map associated with the reflexive sheaf $\mathscr{L}_i$. We have seen above that this finishes the proof of Theorem 7.1. □

Part III. Applications

8. C-SEMITORIC PAIRS

We argue that quotients of semitoric varieties should be seen as C-analogoues of the tori and semitoric varieties that appear in the classic Albanese construction. Before stating our main result on the existence of an Albanese for a C-pair, we define and discuss the relevant notion precisely.

Definition 8.1 (C-semitoric pairs)**.** *A* C-semitoric pair *is a C-pair (X, D) such that there exists a semitoric variety $A^\circ \subset A$, a finite group G acting on (A, Δ_A), and a C-isomorphism of the form*

$$(X, D) \cong (A, \Delta_A) \big/ G \tag{8.1.1}$$

An isomorphism as in (8.1.1) is called a presentation *of the C-semitoric pair.*

The choice of a presentation is not part of the data that defines a C-semitoric pair.

Remark 8.2. If a C-pair (X, D) is C-semitoric, then it is (X, D) is locally uniformizable, [KR24a, Def. 2.32] and X is compact Kähler, cf. [NW14, Prop. 5.3.5].

Example 8.3. The C-pair

$$\left(\mathbb{P}^1, \tfrac{1}{2} \cdot \{0\} + \tfrac{1}{2} \cdot \{1\} + \tfrac{1}{2} \cdot \{2\} + \tfrac{1}{2} \cdot \{\infty\}\right)$$

is C-semitoric.

It is perhaps not obvious from the outset that "C-semitoric pair" is a meaningful notion. In particular, it is probably not clear that morphisms between the open parts of C-semitoric pairs have anything to do with the groups that define the semitoric structures of domain and target. Here, we would like to make the point that *quasi-algebraic* C-morphisms of C-semitoric pairs do indeed come from group morphisms, and therefore respect the structure in a meaningful way. We see this as a strong indication that C-semitoric pairs are relevant objects to consider.

Theorem 8.4 (Morphisms between C-semitoric pairs)**.** *Let (X_1, D_{X_1}) and (X_2, D_{X_2}) be two C-semitoric pairs with presentations*

$$(X_1, D_{X_1}) \cong \big(A_1, \Delta_{A_1}\big) \big/ G_1 \quad \textit{and} \quad (X_2, D_{X_2}) \cong \big(A_2, \Delta_{A_2}\big) \big/ G_2.$$

Given any quasi-algebraic C-morphism $\varphi^\circ : (X_1^\circ, D_{X_1}^\circ) \to (X_2^\circ, D_{X_2}^\circ)$, there exists a semitoric variety $B^\circ \subset B$ and a commutative diagram of the following form,

(8.4.1)
$$\begin{tikzcd}
B^\circ \arrow[r, two heads, "\psi^\circ\text{, quasi-algebraic}", "\text{étale cover}"'] \arrow[d, "\Phi^\circ\text{, quasi-algebraic}"'] & A_1^\circ \arrow[r, two heads, "q_1^\circ\text{, quotient}"] & X_1^\circ \arrow[d, "\varphi^\circ"] \\
A_2^\circ \arrow[r, equal] & A_2^\circ \arrow[r, two heads, "q_2^\circ\text{, quotient}"'] & X_2^\circ.
\end{tikzcd}$$

Remark 8.5 (Quasi-algebraic maps between semitori). Recall Proposition 3.18: if we choose points $0_{B^\circ} \in B^\circ$ and $0_{A_2^\circ} \in A_2^\circ$ to equip B° and A_2° with Lie group structures, then Φ° can be written as a Lie group morphism composed with a translation.

As an immediate corollary to Remark 8.5, we note that quasi-algebraic morphisms of C-semitoric pairs enjoy many of the special properties known for Lie group morphisms. The following corollary lists a few of them.

Corollary 8.6 (Description of morphisms between C-semitoric pairs)**.** *The following holds in the setting of Theorem 8.4.*

(8.6.1) The fibres of φ° are of pure dimension.
(8.6.2) Any two non-empty fibres of φ° are of the same dimension.
(8.6.3) If φ° is quasi-finite, then it is finite. □

8.1. **Proof of Theorem 8.4.** We maintain notation and assumptions of Theorem 8.4 in the present section. To begin, choose a component

$$C^\circ \subseteq \text{normalisation of } A_1^\circ \times_{X_2^\circ} A_2^\circ.$$

The natural morphism $\beta^\circ : C^\circ \twoheadrightarrow A_1^\circ$ is finite. By the analytic version of "Zariski's main theorem in the form of Grothendieck", [DG94, Thm. 3.4], there exists a unique normal compactification $C^\circ \subset C$ where β° extends to a finite morphism $\beta : C \to A_1$. An elementary computation shows that the natural morphism $\eta^\circ : C^\circ \to A_2^\circ$ is quasi-algebraic for this compactification, so that we obtain the following diagram,

(8.7.1)
$$\begin{tikzcd}
C \arrow[d, two heads, "\beta\text{, finite}"'] \arrow[rrr, dashed, bend left, "\eta"] & C^\circ \arrow[l, hook'] \arrow[r, "\eta^\circ"'] \arrow[d, two heads, "\beta^\circ\text{, finite}"'] & A_2^\circ \arrow[r, hook] \arrow[d, equal] & A_2 \arrow[d, equal] \\
A_1 \arrow[d, two heads, "\text{quotient}"'] & A_1^\circ \arrow[l, hook'] \arrow[d, two heads, "q_1^\circ\text{, quotient}"'] & A_2^\circ \arrow[r, hook] \arrow[d, two heads, "q_2^\circ\text{, quotient}"] & A_2 \arrow[d, two heads, "\text{quotient}"] \\
X_1 \arrow[rrr, dashed, bend right, "\varphi"'] & X_1^\circ \arrow[l, hook'] \arrow[r, "\varphi^\circ"] & X_2^\circ \arrow[r, hook] & X_2.
\end{tikzcd}$$

Step 1: Analysis of β. The morphism β is a cover for the logarithmic C-pair (A_1, Δ_{A_1}). Recall from [KR24a, Obs. 3.16] that the associated C-cotangent sheaf equals

$$\Omega^{[1]}_{(A_1,\Delta_{A_1},\beta)} = \beta^* \Omega^1_{A_1}\big(\log \Delta_{A_1}\big). \tag{8.7.2}$$

In particular, we find that the composed pull-back morphism

$$\mathrm{d}\,\beta : H^0\big(A_1,\ \Omega^1_{A_1}(\log \Delta_{A_1})\big) \to H^0\big(C,\ \Omega^{[1]}_C(\log \Delta_C)\big)$$

takes its image in $H^0\big(C,\ \Omega^{[1]}_{(A_1,\Delta_{A_1},\beta)}\big)$. The universal property of the adapted Albanese for the adapted cover β, as specified in Item (5.2.2) of Definition 5.2, will therefore apply to give a factorization

$$\begin{tikzcd} C^\circ \arrow[r, "\mathrm{alb}(A_1,\Delta_{A_1},\beta)^\circ"'] \arrow[rr, bend left, two heads, "\beta^\circ"] & \mathrm{Alb}(A_1, \Delta_{A_1}, \beta)^\circ \arrow[r, two heads, "\psi^\circ"'] & A_1^\circ, \end{tikzcd}$$

where the morphisms β° and $\mathrm{alb}(\ldots)^\circ$ are quasi-algebraic. By Lemma 2.4, then so is the morphism ψ°. We claim that the surjection ψ° is also finite, and hence by Corollary 3.19 an étale cover. Equivalently, we claim that $\mathrm{Alb}(A_1, \Delta_{A_1}, \beta)^\circ \leq \dim A_1^\circ$. But

$$\begin{aligned} \dim \mathrm{Alb}(A_1, \Delta_{A_1}, \beta)^\circ &\leq h^0\big(C,\ \Omega^{[1]}_{(A_1,\Delta_{A_1},\beta)}\big) && \text{Proposition 5.5} \\ &= h^0\big(C,\ \beta^* \Omega^1_{A_1}(\log \Delta_{A_1})\big) && \text{(8.7.2)} \\ &= h^0\big(C,\ \mathscr{O}_C^{\oplus \dim A_1^\circ}\big) = \dim A_1^\circ && \text{Proposition 3.15.} \end{aligned}$$

Step 2: Analysis of η. Recall [KR24a, Obs. 12.10], which implies that the morphisms $q^\circ_\bullet$ of Diagram (8.7.1) are adapted for $(X^\circ_\bullet, D^\circ_\bullet)$ and that the C-cotangent sheaves equal

$$\Omega^{[1]}_{(X^\circ_\bullet, D^\circ_\bullet, q^\circ_\bullet)} = \Omega^1_{A^\circ_\bullet}. \tag{8.7.3}$$

Along similar lines, [KR24a, Obs. 4.15] implies that the morphism $q^\circ_1 \circ \beta^\circ$ is adapted for the pair (X°_1, D°_1), and that

$$\Omega^{[1]}_{(X^\circ_1, D^\circ_1, q^\circ_1 \circ \beta^\circ)} = (\beta^\circ)^{[*]} \Omega^{[1]}_{(X^\circ_1, D^\circ_1, q^\circ_1)} \overset{(8.7.3)}{=} (\beta^\circ)^* \Omega^1_{A^\circ_1}. \tag{8.7.4}$$

The assumption that φ° is a C-morphism implies η° admits pull-back of adapted reflexive differentials,

$$d\eta^\circ : (\eta^\circ)^* \Omega^{[1]}_{(X^\circ_2, D^\circ_2, q^\circ_2)} \to \Omega^{[1]}_{(X^\circ_1, D^\circ_1, q^\circ_1 \circ \beta^\circ)},$$

where

$$\Omega^{[1]}_{(X^\circ_2, D^\circ_2, q^\circ_2)} \overset{(8.7.3)}{=} \Omega^1_{A^\circ_2} \quad \text{and} \quad \Omega^{[1]}_{(X^\circ_1, D^\circ_1, q^\circ_1 \circ \beta^\circ)} \overset{(8.7.4)}{=} (\beta^\circ)^* \Omega^1_{A^\circ_1}.$$

In particular, we find that the composed pull-back morphism

$$\mathrm{d}\,\eta : H^0\big(A_2,\ \Omega^1_{A_2}(\log \Delta_{A_2})\big) \to H^0\big(C,\ \Omega^{[1]}_C(\log \Delta_C)\big)$$

takes its image in

$$H^0\big(C,\ \Omega^{[1]}_C(\log \Delta_C)\big) = H^0\big(C,\ \Omega^{[1]}_{(A_1,\Delta_{A_1},\beta)}\big).$$

As above, the universal property of the adapted Albanese will therefore apply to give a factorization

$$\begin{tikzcd} C^\circ \arrow[r, "\mathrm{alb}(A_1,\Delta_{A_1},\beta)^\circ"'] \arrow[rr, bend left, "\eta^\circ"] & \mathrm{Alb}(A_1, \Delta_{A_1}, \beta)^\circ \arrow[r, "\Phi^\circ"'] & A_2^\circ, \end{tikzcd}$$

where Φ° is quasi-algebraic.

Step 3: Summary. We have seen in Steps 1 and 2 that β° and η° both factor via $\operatorname{alb}(A_1, \Delta_{A_1}, \beta)^\circ$. The following diagram summarizes the situation,

$$\begin{tikzcd}
C^\circ \arrow[r, "\operatorname{alb}(A_1{,}\Delta_{A_1}{,}\beta)^\circ"] & \operatorname{Alb}(A_1, \Delta_{A_1}, \beta)^\circ \arrow[r, two heads, "\psi^\circ\text{, étale}"] \arrow[d, "\Phi^\circ\text{, quasi-algebraic}"'] & A_1^\circ \arrow[r, two heads, "q_1^\circ\text{, quotient}"] & X_1^\circ \arrow[d, "\varphi^\circ"] \\
 & A_2^\circ \arrow[r, equal] & A_2^\circ \arrow[r, two heads, "q_2^\circ\text{, quotient}"] & X_2^\circ.
\end{tikzcd}$$

The proof of Theorem 8.4 is then finished once we set $B^\circ := \operatorname{Alb}(A_1, \Delta_{A_1}, \beta)^\circ$. □

9. The Albanese of a C-pair with bounded irregularity

9.1. **Existence of the Albanese in case of bounded irregularity.** With all the necessary preparation at hand, the main result on the existence of an Albanese of a C-pair is now formulated as follows.

Definition 9.1 (The Albanese of a C-pair)**.** *Let (X, D) be a C-pair where X is compact. An Albanese of (X, D) is a C-semitoric pair $\big(\operatorname{Alb}(X, D), \Delta_{\operatorname{Alb}(X,D)}\big)$ and a quasi-algebraic C-morphism*

$$\operatorname{alb}(X, D)^\circ : (X^\circ, D^\circ) \to \big(\operatorname{Alb}^\circ(X, D), \Delta^\circ_{\operatorname{Alb}(X,D)}\big)$$

such that the following holds: If (S, Δ_S), $s \in S^\circ$ is any other C-semitoric pair and if $s^\circ : (X^\circ, D^\circ) \to (S^\circ, \Delta_S^\circ)$ is any quasi-algebraic C-morphism, then s° factors uniquely as

$$\begin{tikzcd}
(X^\circ, D^\circ) \arrow[r, "\operatorname{alb}(X{,}D)^\circ"'] \arrow[rr, bend left, "s^\circ"] & \big(\operatorname{Alb}_x^\circ(X, D), \Delta^\circ_{\operatorname{Alb}(X,D)}\big) \arrow[r, "\exists!c^\circ\text{, quasi-algebraic}"'] & (S^\circ, \Delta_S^\circ).
\end{tikzcd}$$

Theorem 9.2 (The Albanese of a C-pair with bounded irregularity)**.** *Let (X, D) be a C-pair where X is compact Kähler. If $q^+_{\operatorname{Alb}}(X, D) < \infty$, then an Albanese of (X, D) exists.*

Theorem 9.2 will be shown in Section 9.3 below.

Remark 9.3 (Special pairs). Recall from Remark 7.4 on page 24 that the assumption $q^+_{\operatorname{Alb}}(X, D) < \infty$ is always satisfied if the C-pair (X, D) is special.

Remark 9.4 (Uniqueness). The universal property implies that $\operatorname{Alb}(X, D)^\circ$ is unique up to unique isomorphism and that $\operatorname{Alb}(X, D)$ is bimeromorphically unique. The universal property also implies that $\dim \operatorname{Alb}(X, D) = q^+_{\operatorname{Alb}}(X, D)$.

As before, we abuse notation and refer to any Albanese of (X, D) as "the Albanese".

9.2. **Non-existence of the Albanese in case of unbounded irregularity.** Before proving of Theorem 9.2, we remark that the assumption $q^+_{\operatorname{Alb}}(X, D) < \infty$ is necessary in the strongest possible sense.

Proposition 9.5 (Non-existence of the Albanese in case of unbounded irregularity)**.** *Let (X, D) be a C-pair where X is compact Kähler. If $q^+_{\operatorname{Alb}}(X, D) = \infty$, then an Albanese of (X, D) cannot possibly exist.*

Proof. We argue by contradiction and assume that there exists a C-semitoric pair $\big(A, \Delta_A\big)$ and a quasi-algebraic C-morphism

$$a^\circ : (X^\circ, D^\circ) \to \big(A^\circ, \Delta_A^\circ\big)$$

that satisfies the universal property of Theorem 9.2. By assumption, there exists a cover $\gamma : \widehat{X} \twoheadrightarrow X$ such that $q_{\operatorname{Alb}}(X, D, \gamma) > \dim A^\circ$. Lemma 5.8 on page 18 allows assuming without loss of generality that γ is Galois with group G. Proposition 5.13 on page 20 yields a diagram

$$\begin{tikzcd}
\widehat{X}^\circ \arrow[r, "\operatorname{alb}^\circ(X,D,\gamma)"] \arrow[d, two heads, "\gamma"'] & \operatorname{Alb}^\circ(X,D,\gamma) \arrow[d, two heads, "\text{quotient}"] \\
X^\circ \arrow[r, "s^\circ"'] & \operatorname{Alb}^\circ(X,D,\gamma) \big/ G
\end{tikzcd} \tag{9.5.1}$$

where s° is a quasi-algebraic morphism of C-pairs,

$$s^\circ : (X^\circ, D^\circ) \to \big(\operatorname{Alb}^\circ(X,D,\gamma), 0\big) \big/ G.$$

By assumption, the C-morphism s° factors via a°, and equips us with a quasi-algebraic morphism of C-pairs,

$$c^\circ : (A^\circ, \Delta_A^\circ) \to \big(\operatorname{Alb}^\circ(X,D,\gamma), 0\big) \big/ G.$$

Observing that domains and target of the C-morphism c° are C-semitoric pairs, Theorem 8.4 yields a semitoric variety $(\check{A}, \Delta_{\check{A}})$ and an extension of Diagram (9.5.1) as follows,

$$\begin{tikzcd}
\check{X}^\circ \arrow[r] \arrow[d, two heads, "\text{finite}"'] & \check{A}^\circ \arrow[r, "\Phi^\circ\text{, quasi-algebraic}"] \arrow[dd, two heads, "\text{quotient} \circ \text{étale}"] & \operatorname{Alb}^\circ(X,D,\gamma) \arrow[d, equal] \\
\widehat{X}^\circ \arrow[rr, "\operatorname{alb}^\circ(X,D,\gamma)"] \arrow[d, two heads, "\gamma"'] & & \operatorname{Alb}^\circ(X,D,\gamma) \arrow[d, two heads, "\text{quotient}"] \\
X^\circ \arrow[r, "a^\circ"] \arrow[rr, bend right, "s^\circ"'] & A^\circ \arrow[r, "c^\circ"] & \operatorname{Alb}^\circ(X,D,\gamma) \big/ G
\end{tikzcd}$$

where Φ° is quasi-algebraic. Since

$$\dim \check{A}^\circ = \dim A^\circ < \dim \operatorname{Alb}(X,D,\gamma)$$

by construction, it is clear that Φ° cannot be surjective. It follows that the image of the Albanese morphism $\operatorname{alb}^\circ(X,D,\gamma)$ is contained in $\operatorname{img} \Phi^\circ \subsetneq \operatorname{Alb}^\circ(X,D,\gamma)$, contradicting the assertion of Proposition 6.5 that the image generates $\operatorname{Alb}(X,D,\gamma)^\circ$ as an Abelian group, once appropriate Lie group structures are chosen. □

9.3. **Proof of Theorem 9.2.** We maintain notation and assumptions of Theorem 9.2. The proof is somewhat long, as it involves the discussion of a fair number of diagrams and references to almost all results obtained so far. For the reader's convenience, we present the argument in four relatively independent steps.

Step 1: Choices and constructions. We consider the set of Galois covers,

$$M := \big\{\delta : \widehat{X}_\delta \twoheadrightarrow X \text{ a Galois cover of } (X,D)\big\}.$$

For every $\delta \in M$, write G_δ for the associated Galois group, write $\widehat{X}_\delta^\circ := \delta^{-1}(X^\circ) \subseteq \widehat{X}_\delta$ and denote the restriction of δ by $\delta^\circ : \widehat{X}_\delta^\circ \to X^\circ$.

Choice 9.6 (Comparison morphism). For every pair of two covers $\delta_1, \delta_2 \in M$ where δ_1 factors via δ_2,

$$\begin{tikzcd}
\widehat{X}_{\delta_1} \arrow[r, two heads, "\exists\, \delta_{12}"'] \arrow[rr, two heads, bend left, "\delta_1"] & \widehat{X}_{\delta_2} \arrow[r, two heads, "\delta_2"'] & X,
\end{tikzcd} \tag{9.6.1}$$

choose one morphism $\delta_{12} : \widehat{X}_{\delta_1} \twoheadrightarrow \widehat{X}_{\delta_2}$ that makes (9.6.1) commute.

Choice 9.7 (Albanese varieties for the covers). For every $\delta \in M$, use Proposition 5.5 and Corollary 3.21 to choose an Albanese $\big(A_\delta, \Delta_{A_\delta}\big)$ of the cover δ, where the G_δ-action extends from A°_δ to A_δ. Denote the associated quasi-algebraic Albanese morphism by

$$\widehat{a}^\circ_\delta : \widehat{X}^\circ_\delta \to A^\circ_\delta.$$

Notation 9.8 ($\mathcal{C}$-semitoric pairs). With the choices above, consider the $\mathcal{C}$-semitoric pairs

$$(B_\delta, \Delta_{B_\delta}) := (A_\delta, \Delta_{A_\delta}) \Big/ G_\delta.$$

Let $a^\circ_\delta : (X^\circ, D^\circ) \to (B^\circ_\delta, \Delta^\circ_{B_\delta})$ be the quasi-algebraic $\mathcal{C}$-morphisms introduced in Proposition 5.13.

In the sequel, we will need to compare the $\mathcal{C}$-semitoric pairs induced by two covers that factor one another. The following reminder summarizes what we already know.

Reminder 9.9 (Comparing covers). Given two covers $\delta_1, \delta_2 \in M$ where δ_1 factors via δ_2, Lemma 5.14 equips us with a commutative diagram

$$\begin{tikzcd}
\widehat{X}^\circ_{\delta_1} \arrow[r, "\widehat{a}^\circ_{\delta_1}"] \arrow[d, two heads, "\delta_{12}"] \arrow[dd, two heads, bend right, "\delta^\circ_1"'] & A^\circ_{\delta_1} \arrow[dr, "{\widehat{q}^\circ_{\delta_1\delta_2}\text{, quasi-algebraic}}"] \arrow[dd, two heads, "{q_{\delta_1}\text{, finite quotient}}"] & \\
\widehat{X}^\circ_{\delta_2} \arrow[rr, "\widehat{a}^\circ_{\delta_2}"] \arrow[d, two heads, "\delta^\circ_2"] & & A^\circ_{\delta_2} \arrow[d, two heads, "{q_{\delta_2}\text{, finite quotient}}"] \\
X^\circ \arrow[r, "a^\circ_{\delta_1}"] \arrow[rr, bend right, "a^\circ_{\delta_2}"'] & B^\circ_{\delta_1} \arrow[r, two heads, "q^\circ_{\delta_1\delta_2}"] & B^\circ_{\delta_2},
\end{tikzcd} \tag{9.9.1}$$

where all morphisms are quasi-algebraic and all morphisms in the bottom row are morphisms of $\mathcal{C}$-pairs, between (X°, D°), $(B^\circ_{\delta_1}, \Delta^\circ_{B_{\delta_1}})$ and $(B^\circ_{\delta_2}, \Delta^\circ_{B_{\delta_2}})$.

Choice 9.10 (Albanese of (X, D)). Consider the numbers

$$n_\delta := \#\text{components in the typical non-empty fibre of } a^\circ_\delta : X^\circ \to B^\circ_\delta,$$
$$n_{\min} := \min\big\{n_\delta \;:\; \delta \in M \text{ and } \dim B_\delta = q^+_{\mathrm{Alb}}(X, D)\big\}$$

and choose one particular cover $\gamma \in M$ such that $\dim B_\gamma = q^+_{\mathrm{Alb}}(X, D)$ and $n_\gamma = n_{\min}$. Once the choice is made, consider the associated $\mathcal{C}$-semitoric pair

$$\big(\mathrm{Alb}(X, D), \Delta_{\mathrm{Alb}(X,D)}\big) := (B_\gamma, \Delta_{B_\gamma})$$

with the associated morphism $\mathrm{alb}(X, D)^\circ := a^\circ_\gamma$. We will show that this is an Albanese of (X, D).

Step 2: First properties of the construction. We need to show that our choice of an Albanese does indeed satisfy the universal properties required by Definition 9.1. To prepare for the proof, we study covers $\delta \in M$ that factor via γ. The following claims show that the $\mathcal{C}$-morphism $q^\circ_{\delta\gamma} : B^\circ_\delta \to B^\circ_\gamma$ of Reminder 9.9 is an isomorphism of $\mathcal{C}$-pairs. The proof makes extensive use of the notation introduced in Reminder 9.9. The reader might wish to write down Diagram (9.9.1) in our particular situation, where δ_1 is replaced by δ and δ_2 is replaced by γ.

Claim 9.11. Assume that a cover $\delta \in M$ factors via γ. Then, the morphism $q^\circ_{\delta\gamma}$ of Reminder 9.9 is finite as a morphism of analytic varieties.

Proof of Claim 9.11. The choices made in Step 2 guarantee that the quasi-algebraic morphism $\widehat{q}^\circ_{\delta\gamma}$ is surjective between semitori of the same dimension. It follows from Corollary 3.19 that $\widehat{q}^\circ_{\delta\gamma}$ is finite. As the induced morphism between (finite) Galois quotients, $q^\circ_{\delta\gamma}$ is then likewise finite. □ (Claim 9.11)

Claim 9.12. Assume that a cover $\delta \in M$ factors via γ. Then, the morphism $q^\circ_{\delta\gamma}$ of Reminder 9.9 is biholomorphic as a morphism of analytic varieties.

Proof of Claim 9.12. If $z \in \operatorname{img} a^\circ_\gamma \subseteq B^\circ_\gamma$ is general, observe that the following two conditions hold.

The morphism $q^\circ_{\delta\gamma}$ is étale over z: We have seen in Proposition 5.13 that z is not contained in the branch locus of the finite quotient map q_γ. In other words, q_γ is étale over z. Corollary 3.19 guarantees that $\widehat{q}^\circ_{\delta\gamma}$ is étale everywhere, so that

$$q_\gamma \circ \widehat{q}^\circ_{\delta\gamma} = q^\circ_{\delta\gamma} \circ q_\delta$$

is étale over z. But then $q^\circ_{\delta\gamma}$ is étale over z.

The set-theoretic fibre $\left(q^\circ_{\delta\gamma}\right)^{-1}(z) \subset B^\circ_\delta$ is connected: This is a direct consequence of Choice 9.10.

Given that the number of fibre components is constant in finite, étale morphisms, we find that the finite morphism $q^\circ_{\delta\gamma}$ has connected fibres. It is hence a one-sheeted analytic covering in the sense of [Rem94, Sect. 14.2]. Together with normality, [Rem94, Prop. 14.7] applies to show that it is indeed biholomorphic. □ (Claim 9.12)

Claim 9.13. Assume that a cover $\delta \in M$ factors via γ. Then, the morphism $q^\circ_{\delta\gamma}$ of Reminder 9.9 is isomorphic as a morphism of C-pairs.

Proof. Using the biholomorphic map $q^\circ_{\delta\gamma}$ to identify the analytic varieties B°_δ and B°_γ, we need to show that the boundary divisors induced by the quotient morphism q_δ and q_γ agree. The construction of categorical C-pair quotients, [KR24a, Cons. 12.4], tells us what the boundaries are: if H_δ is any prime divisor in B°_δ and if we choose prime divisors in the preimages spaces,

$$\begin{aligned} H_\gamma &= \left(\left(q^\circ_{\delta\gamma}\right)^{-1}\right)^* H_\delta && \text{prime divisor in } B^\circ_\gamma \\ \widehat{H}_\gamma &\le \left(q_\gamma\right)^* H_\gamma && \text{prime divisor in } A^\circ_\gamma \\ \widehat{H}_\delta &\le \left(\widehat{q}^\circ_{\delta\gamma}\right)^* H_\gamma && \text{prime divisor in } A^\circ_\delta, \end{aligned}$$

then

$$\operatorname{mult}_{C,H_\delta} \Delta^\circ_{B_\delta} = \operatorname{mult}_{\widehat{H}_\delta} (q_\delta)^* H_\delta \quad \text{and} \quad \operatorname{mult}_{C,H_\gamma} \Delta^\circ_{B_\gamma} = \operatorname{mult}_{\widehat{H}_\gamma} (q_\delta)^* H_\gamma.$$

But these two numbers agree, given that $q^\circ_{\delta\gamma}$ and $\widehat{q}^\circ_{\delta\gamma}$ are étale. □ (Claim 9.13)

Step 4: Universal property. We will now show that the constructions of the previous steps satisfy the universal property spelled out in Definition 9.1. We fix the setting for the remainder of the present proof.

Setting 9.14 (Universal property). Let (B, Δ_B) be a C-semitoric pair and assume that a quasi-algebraic C-morphism $a^\circ : (X^\circ, D^\circ) \to (B^\circ, \Delta^\circ_B)$ is given. Let

$$(B, \Delta_B) \cong (A, \Delta_A) \big/ G$$

be a presentation of the C-semitoric pair, with quotient morphism $q : A \twoheadrightarrow B$.

We need to show that the C-morphism a° factors via a°_γ uniquely. In other words, we need to find a quasi-algebraic C-morphism c° fitting into a commutative diagram

(9.15.1)
$$\begin{tikzcd}
(X^\circ, D^\circ) \arrow[r, "a^\circ_\gamma"'] \arrow[rr, bend left, "a^\circ"] & \bigl(B^\circ_\gamma, \Delta^\circ_{B_\gamma}\bigr) \arrow[r, "\exists! c^\circ"'] & (B^\circ, \Delta^\circ_B)
\end{tikzcd}$$

and prove that c° is unique with this property.

Step 4a: Existence of a factorization. Maintaining Setting 9.14, we show that there exists one quasi-algebraic C-morphism $c^\circ : \bigl(B^\circ_\gamma, \Delta^\circ_{B_\gamma}\bigr) \to (B^\circ, \Delta^\circ_B)$ that makes Diagram (9.15.1) commute.

Construction 9.16 (Fibre product). Choose a component of the normalized fibre product

$$\widehat{X}_\rho \subseteq \text{normalization of } A \times_B X.$$

Denote the projection morphisms and their restrictions as follows,

(9.16.1)
$$\begin{tikzcd}
\widehat{X}_\rho \arrow[d, two heads, "\rho"'] & \supseteq & \widehat{X}^\circ_\rho \arrow[r, "\widehat{a}^\circ"] \arrow[d, two heads, "\rho^\circ"'] & A^\circ \arrow[d, two heads, "q^\circ"] & \subseteq & A \arrow[d, two heads, "{q, \text{ quotient}}"] \\
X & \supseteq & X^\circ \arrow[r, "a^\circ"'] & B^\circ & \subseteq & B.
\end{tikzcd}$$

Observation 9.17 (Group actions in (9.16.1)). The group G acts on the fibre product $A \times_B X$ and on its normalization. The stabilizer of the component $\widehat{X}_\rho$,

$$H := \operatorname{Stab}\bigl(\widehat{X}_\rho\bigr) \subseteq G,$$

acts on $\widehat{X}_\rho$, and $\rho : \widehat{X}_\rho \twoheadrightarrow X$ is the quotient map of this action. The projection map ρ is therefore Galois. In other words, $\rho \in M$. The Galois group is the quotient of H by the ineffectivity,

$$G_\rho = {}^{H}\big/_{\ker\left(H \to \operatorname{Aut} \widehat{X}_\rho\right)}.$$

The projection map $\widehat{a}^\circ$ is equivariant with respect to the action of H on $\widehat{X}^\circ_\rho$ and on A°.

Observation 9.18 (Factorization via the Albanese of the cover). We have seen in [KR24a, Obs. 12.10] that the quotient morphism q° is an adapted cover for the pair $(B^\circ, \Delta^\circ_B)$ and that the adapted differentials are described as

$$\Omega^{[1]}_{(B^\circ, \Delta^\circ_B, q^\circ)} = \Omega^1_{(B^\circ, \Delta^\circ_B, q^\circ)} = \Omega^1_{A^\circ}.$$

The assumption that b° is a C-morphism guarantees by definition that Diagram (9.16.1) admits pull-back of adapted reflexive differentials. In other words: The composed pull-back morphism

$$(\widehat{a}^\circ)^* \Omega^1_{A^\circ} = (\widehat{a}^\circ)^* \Omega^1_{(B^\circ, B^\circ_S, q^\circ)} \xrightarrow{d\,\widehat{a}^\circ} \Omega^1_{\widehat{X}^\circ_\rho} \to \Omega^{[1]}_{\widehat{X}^\circ_\rho}$$

takes its image in the subsheaf $\Omega^{[1]}_{(X^\circ, D^\circ, \rho)} \subseteq \Omega^{[1]}_{\widehat{X}^\circ_\rho}$. The universal property of the Albanese for the cover ρ, Item (5.2.2) of Definition 5.2, will then guarantee that $\widehat{a}^\circ$ factors as

(9.18.1)
$$\begin{tikzcd}
\widehat{X}^\circ_\rho \arrow[r, "\widehat{a}^\circ_\rho"'] \arrow[rr, bend left, "\widehat{a}^\circ"] & A^\circ_\rho \arrow[r, "\widehat{s}^\circ"'] & A^\circ,
\end{tikzcd}$$

where $\widehat{s}^\circ$ is a quasi-algebraic morphism of semitori. There is more that we can say: G_ρ is the Galois group of the covering morphism ρ and therefore acts on A_ρ, the Albanese of the cover ρ. The group G_ρ is a quotient of H, and the universal property of the Albanese A_ρ guarantees that the map $\widehat{a}^\circ$ is equivariant with respect to the H-action.

Claim 9.19 (Extension of (9.16.1) and (9.18.1)). There exists a commutative diagram of morphisms between analytic varieties,

$$\begin{tikzcd}
\widehat{X}^\circ_\rho \arrow[r, "\widehat{a}^\circ_\rho"'] \arrow[d, two heads, "\rho^\circ"'] \arrow[rrr, bend left, "\widehat{a}^\circ"] & A^\circ_\rho \arrow[r, "\widehat{s}^\circ"'] \arrow[d, two heads, "{q_\rho, \text{ quot. by } G_\rho}"] & A^\circ \arrow[r, equal] \arrow[d, two heads, "{q'_\rho, \text{ quot. by } G_\rho}"] & A^\circ \arrow[d, two heads, "{q^\circ, \text{ quot. by } G}"] \\
X^\circ \arrow[r, "a^\circ_\rho"] \arrow[rrr, bend right, "a^\circ"'] & B^\circ_\rho \arrow[r, "s^\circ"] & A^\circ\big/H \arrow[r, "\beta^\circ"] & A^\circ\big/G,
\end{tikzcd} \tag{9.19.1}$$

where all morphisms in the bottom row are quasi-algebraic C-morphisms between (X°, D°) and the C-pairs

$$(B^\circ_\rho, \Delta^\circ_{B_\rho}) = (A^\circ_\rho, 0)\big/G_\rho = (A^\circ_\rho, 0)\big/H, \quad (A^\circ, 0)\big/H \quad \text{and} \quad (B^\circ, \Delta^\circ_B) = (A^\circ, 0)\big/G.$$

Proof of Claim 9.19. Given that A° is a semitorus hence smooth, recall from [KR24a, Ex. 8.6] that the H-equivariant morphism $\widehat{s}^\circ$ is a C-morphism between the trivial pairs $(A^\circ_\rho, 0)$ and $(A^\circ, 0)$. Let s° be induced by the C-morphism between the quotient pairs

$$(B^\circ_\rho, \Delta^\circ_{B_\rho}) = (A^\circ_\rho, 0)\big/H \quad \text{and} \quad (A^\circ, 0)\big/H,$$

as discussed in [KR24a, Prop. 12.7]. This morphism makes the middle square in (9.19.1) commute.

Next, we need to define the morphism β°. For that, review the definition of categorical quotients of C-pairs, [KR24a, Def. 12.3]. The definition guarantees on the one hand that q'_ρ and q° are C-morphisms between the C-pairs

$$(A^\circ, 0), \quad (A^\circ, 0)\big/H \quad \text{and} \quad (A^\circ, 0)\big/G.$$

Given that q° is constant on the fibres of q'_ρ, the definition also says that q° factors via q'_ρ, as required. The induced C-morphism β° makes the right square in (9.19.1) commute.

It remains to show that $a^\circ = \beta^\circ \circ s^\circ \circ a^\circ_\rho$. That, however, follows from the equality $q^\circ \circ \widehat{a}^\circ = a^\circ \circ \rho^\circ$ given by (9.16.1), using that ρ° is surjective. □ (Claim 9.19)

Assumption w.l.o.g. 9.20 (Factorization of γ). The C-pair $\big(B^\circ_\gamma, \Delta^\circ_{B_\gamma}\big)$ and the morphism a°_γ have been defined above using the cover γ, but we have seen in Claim 9.13 that they can equally be defined by any cover that factors via γ. Replacing $\widehat{X}_\gamma$ by the Galois closure of a suitable normalized fibre product, we may therefore assume without loss of generality that γ factors via ρ,

$$\begin{tikzcd}
\widehat{X}_\gamma \arrow[r, two heads] \arrow[rr, two heads, bend left, "\gamma"] & \widehat{X}_\rho \arrow[r, two heads, "\rho"'] & X.
\end{tikzcd}$$

With Assumption 9.20 in place, the existence of a factorization is now immediate. Reminder 9.9 decomposes the left square in (9.19.1) as follows,

$$\begin{tikzcd}
\widehat{X}^\circ_\gamma \arrow[r, "\widehat{a}^\circ_\gamma"'] \arrow[d, two heads, "\gamma^\circ"'] \arrow[rr, two heads, bend left, "\widehat{a}^\circ_\rho"] & A^\circ_\gamma \arrow[r, two heads, "\widehat{q}^\circ_{\gamma\rho}"'] \arrow[d, two heads, "q^\circ_\gamma"] & A^\circ_\rho \arrow[d, two heads, "q^\circ_\rho"] \\
X^\circ \arrow[r, "a^\circ_\gamma"] \arrow[rr, two heads, bend right, "a^\circ_\rho"'] & B^\circ_\gamma \arrow[r, two heads, "q^\circ_{\gamma\delta}"] & B^\circ_\rho
\end{tikzcd}$$

where all morphisms are quasi-algebraic and all morphisms in the bottom row are morphisms of C-pairs, between (X°, D°), $(B^\circ_\gamma, \Delta^\circ_{B_\gamma})$ and $(B^\circ_\rho, \Delta^\circ_{B_\rho})$. We can then set

$$c^\circ := \beta^\circ \circ s^\circ \circ q^\circ_{\gamma\delta}.$$

A factorization is thus found.

Step 4b: Uniqueness of the factorization. Maintain Setting 9.14 and assume that there are two quasi-algebraic C-morphisms that makes Diagram (9.15.1) commute,

$$\begin{tikzcd}
(X^\circ, D^\circ) \arrow[r, "a^\circ_\gamma"'] \arrow[rr, bend left, "a^\circ"] & \bigl(B^\circ_\gamma, \Delta^\circ_{B_\gamma}\bigr) \arrow[r, "\exists\, c^\circ_1,\, c^\circ_2"'] & (B^\circ, \Delta^\circ_B).
\end{tikzcd}$$

We need to show that the two morphisms are equal, $c^\circ_1 = c^\circ_2$.

Construction 9.21 (Lifting $c^\circ_\bullet$ to Lie group morphisms). Theorem 8.4 equips us with semitoric varieties $\widehat{A}^\circ_{\gamma,\bullet} \subset \widehat{A}_{\gamma,\bullet}$, quasi-algebraic isogenies $i^\circ_\bullet : \widehat{A}^\circ_{\gamma,\bullet} \twoheadrightarrow A^\circ_\gamma$ and quasi-algebraic Lie group morphisms $\Phi^\circ_\bullet : \widehat{A}^\circ_{\gamma,\bullet} \to A^\circ$ forming commutative diagrams as in (8.4.1),

$$\begin{tikzcd}
\widehat{A}^\circ_{\gamma,\bullet} \arrow[r, "{\Phi^\circ_\bullet,\ \text{quasi-algebraic}}"] \arrow[d, two heads, "{i^\circ_\bullet,\ \text{quasi-algebraic and étale}}"'] & A^\circ \arrow[d, equal] \\
A^\circ_\gamma \arrow[d, two heads, "{q^\circ_\gamma,\ \text{quotient}}"'] & A^\circ \arrow[d, two heads, "{q^\circ,\ \text{quotient}}"] \\
B^\circ_\gamma \arrow[r, "c^\circ_\bullet"'] & B^\circ.
\end{tikzcd}$$

Blowing up in a left-invariant manner, we may assume without loss of generality that the $i^\circ_\bullet$ extend to morphisms $i_\bullet : \widehat{A}_{\gamma,\bullet} \twoheadrightarrow A_\gamma$.

Define a semitoric variety $\widehat{A}^\circ_\gamma \subset \widehat{A}_\gamma$ by choosing a suitable strong resolution of a component of the fibre product $\widehat{A}_{\gamma,1} \times_{A_\gamma} \widehat{A}_{\gamma,2}$. The natural maps

$$\widehat{A}^\circ_\gamma \twoheadrightarrow \widehat{A}^\circ_{\gamma,\bullet} \twoheadrightarrow A^\circ_\gamma$$

are then quasi-algebraic and étale. Compose the projection maps $\widehat{A}^\circ_\gamma \to \widehat{A}^\circ_{\gamma,\bullet}$ with $\Phi^\circ_\bullet$ to obtain two quasi-algebraic morphisms between semitori, $\widehat{\varphi}^\circ_\bullet : \widehat{A}^\circ_\gamma \to A^\circ$, each making the following diagram commute,

$$\begin{tikzcd}
\widehat{A}^\circ_\gamma \arrow[r, "{\widehat{\varphi}^\circ_\bullet,\ \text{quasi-algebraic}}"] \arrow[d, two heads, "{i^\circ,\ \text{quasi-algebraic étale}}"'] & A^\circ \arrow[d, equal] \\
A^\circ_\gamma \arrow[d, two heads, "{q^\circ_\gamma,\ \text{quotient}}"'] & A^\circ \arrow[d, two heads, "{q^\circ,\ \text{quotient}}"] \\
B^\circ_\gamma \arrow[r, "c^\circ_\bullet"'] & B^\circ.
\end{tikzcd} \tag{9.21.1}$$

Construction 9.22 (Dominating γ). Continuing Construction 9.21, choose a component of the normalized fibre product $\widehat{A}_\gamma \times_{A_\gamma} \widehat{X}_\gamma$ and let $\widehat{X}_\delta$ be the Galois closure of that component over X. We obtain a Galois cover $\delta : \widehat{X}_{\delta} \twoheadrightarrow X$ and a commutative diagram of quasi-algebraic morphisms as follows,

(9.22.1)
$$\begin{tikzcd}
\widehat{X}^{\circ}_{\delta} \arrow[r, "\mu^{\circ}"] \arrow[d, two heads] \arrow[dd, bend right=40, two heads, "\delta^{\circ}\text{, Galois cover}"'] & \widehat{A}^{\circ}_{\gamma} \arrow[d, two heads, "i^{\circ}\text{, quasi-algebraic étale}"] \\
\widehat{X}^{\circ}_{\gamma} \arrow[r, "\widehat{a}^{\circ}_{\gamma}"] \arrow[d, two heads, "\gamma^{\circ}"] & A^{\circ}_{\gamma} \arrow[d, two heads, "q^{\circ}_{\gamma}\text{, quotient}"] \\
X^{\circ} \arrow[r, "a^{\circ}_{\gamma}"'] & B^{\circ}_{\gamma}.
\end{tikzcd}$$

Observation 9.23 (Factorization via the Albanese of the cover). In analogy to Observation 9.18, recall from [KR24a, Obs. 12.10] that the quotient morphism q°_{γ} is an adapted cover for the pair $(B^{\circ}_{\gamma}, \Delta^{\circ}_{B_{\gamma}})$. Since i° is étale, the adapted differentials are described as

$$\Omega^{[1]}_{(B^{\circ}_{\gamma}, \Delta^{\circ}_{B_{\gamma}}, q^{\circ}_{\gamma} \circ i^{\circ})} = \Omega^{1}_{\widehat{A}^{\circ}_{\gamma}}.$$

Since a°_{γ} is a $\mathcal{C}$-morphism, Diagram (9.22.1) admits pull-back of adapted reflexive differentials. The composed pull-back morphism

$$(\mu^{\circ})^{*}\Omega^{1}_{\widehat{A}^{\circ}_{\gamma}} = (\mu^{\circ})^{*}\Omega^{1}_{(B^{\circ}_{\gamma}, \Delta^{\circ}_{B_{\gamma}}, q^{\circ}_{\gamma} \circ i^{\circ})} \xrightarrow{d\mu^{\circ}} \Omega^{1}_{\widehat{X}^{\circ}_{\delta}} \to \Omega^{[1]}_{\widehat{X}^{\circ}_{\delta}}$$

therefore takes its image in the subsheaf $\Omega^{[1]}_{(X^{\circ}, D^{\circ}, \delta^{\circ})} \subseteq \Omega^{[1]}_{\widehat{X}^{\circ}_{\delta}}$. The universal property of the Albanese for the cover δ°, Item (5.2.2) of Definition 5.2, will then guarantee that $\widehat{a}^{\circ}$ factors as follows,

(9.23.1)
$$\begin{tikzcd}
\widehat{X}^{\circ}_{\rho} \arrow[r, "\widehat{a}^{\circ}_{\delta}"'] \arrow[rr, bend left=20, "\mu^{\circ}"] & A^{\circ}_{\delta} \arrow[r, "\nu^{\circ}\text{, quasi-algebraic}"'] & \widehat{A}^{\circ}_{\gamma}.
\end{tikzcd}$$

Summary 9.24. Combining (9.21.1), (9.22.1) and (9.23.1), the following two diagrams summarize the constructions obtained so far,

$$\begin{tikzcd}
\widehat{X}^{\circ}_{\delta} \arrow[r, "\widehat{a}^{\circ}_{\delta}"'] \arrow[rr, bend left=15, "\mu^{\circ}"] \arrow[d, two heads] \arrow[dd, bend right=40, two heads, "\delta^{\circ}"'] & A^{\circ}_{\delta} \arrow[r, "\nu^{\circ}\text{, quasi-algebraic}"'] \arrow[d, two heads, "q^{\circ}_{\delta\gamma}"] & \widehat{A}^{\circ}_{\gamma} \arrow[r, "\widehat{\varphi}^{\circ}_{\bullet}\text{, quasi-algebraic}"] \arrow[d, two heads, "i^{\circ}\text{, quasi-algebraic étale}"] & A^{\circ} \arrow[d, equal] \\
\widehat{X}^{\circ}_{\gamma} \arrow[r, "\widehat{a}^{\circ}_{\gamma}"] \arrow[d, two heads, "\gamma^{\circ}"] & A^{\circ}_{\gamma} \arrow[r, equal] \arrow[d, two heads, "q^{\circ}_{\gamma}"] & A^{\circ}_{\gamma} \arrow[d, two heads, "q^{\circ}_{\gamma}"] & A^{\circ} \arrow[d, two heads, "q^{\circ}"] \\
X^{\circ} \arrow[r, "a^{\circ}_{\gamma}"] \arrow[rrr, bend right=15, "a^{\circ}"'] & B^{\circ}_{\gamma} \arrow[r, equal] & B^{\circ}_{\gamma} \arrow[r, "c^{\circ}_{\bullet}"] & B^{\circ}.
\end{tikzcd}$$

Setting $\widehat{c}^\circ_\bullet := \widehat{\varphi}^\circ_\bullet \circ \nu^\circ$, we are interested in the subdiagrams

(9.24.1)
$$\begin{tikzcd}
\widehat{X}^\circ_\delta \arrow[r, "\widehat{a}^\circ_\delta"] \arrow[d, two heads, "\delta^\circ"'] & A^\circ_\delta \arrow[r, "{\widehat{c}^\circ_\bullet,\ \text{quasi-algebraic}}"] \arrow[d, two heads, "q^\circ_\gamma \circ q^\circ_{\delta\gamma}"] & A^\circ \arrow[d, two heads, "q^\circ"] \\
X^\circ \arrow[r, "a^\circ_\gamma"] \arrow[rr, bend right, "a^\circ"'] & B^\circ_\gamma \arrow[r, "c^\circ_\bullet"] & B^\circ .
\end{tikzcd}$$

Claim 9.25 (Aligning the $\widehat{c}^\circ_\bullet$). Recalling that G is the Galois group of the morphism q, there exists an element $g \in G$ such that $\widehat{c}^\circ_1 \circ \widehat{a}^\circ_\delta = g \circ \widehat{c}^\circ_2 \circ \widehat{a}^\circ_\delta$.

Proof of Claim 9.25. For every point of $x \in \widehat{X}^\circ_\delta$, commutativity of (9.24.1) guarantees that the image points $(\widehat{c}^\circ_\bullet \circ \widehat{a}^\circ_\delta)(x)$ are contained in the same fibre of the Galois morphism q°. Accordingly, there exists an element $g_x \in G$ such that

(9.25.1)
$$\widehat{c}^\circ_1 \circ \widehat{a}^\circ_\delta(x) = g_x \circ \widehat{c}^\circ_2 \circ \widehat{a}^\circ_\delta(x).$$

But since G is finite, there exists one $g \in G$ such that (9.25.1) holds for every $x \in \widehat{X}^\circ_\delta$. □ (Claim 9.25)

To conclude, choose one Galois element $g \in G$ as in Claim 9.25. Choose a point $x \in \widehat{X}^\circ_\delta$ and use

$$0_{A^\circ_\delta} := \widehat{a}^\circ_\delta(x) \quad \text{and} \quad 0_{A^\circ} := \widehat{c}^\circ_1(0_{A^\circ_\delta})$$

to equip A°_δ and A° with Lie group structures. With these structures, Proposition 3.18 guarantees that

$$\widehat{c}^\circ_1 : A^\circ_\delta \to A^\circ, \quad g \circ \widehat{c}^\circ_2 : A^\circ_\delta \to A^\circ, \quad \text{and} \quad g : A^\circ_\delta \to A^\circ_\delta$$

are Lie group morphisms, so that

$$\operatorname{img} \widehat{a}^\circ_\delta \subseteq \ker\bigl(\widehat{c}_1 - g \circ \widehat{c}_2\bigr) \subseteq A^\circ_\delta.$$

Recalling from Proposition 5.5 that $\operatorname{img} \widehat{a}^\circ_\delta$ generates A°_δ as a group, we find that $\widehat{c}_1 = g \circ \widehat{c}_2$. Commutativity of (9.21.1) and surjectivity of q°_δ then show that $c^\circ_1 = c^\circ_2$, as required to finish the proof of Theorem 9.2. □

10. Problems and open questions

10.1. **A weak Albanese for arbitrary C-pairs.** If (X, D) is a C-pair $q^+_{\mathrm{Alb}}(X, D) = \infty$, then we have seen in Proposition 9.5 that an Albanese in the sense of Definition 9.1 cannot possibly exist. While it might be possible to define a meaningful Albanese as an ind-variety or as a (yet to be defined) ind-pair, we are convinced that a weak version of the Albanese does exist within the world of classical C-pairs. For many practical purposes, the following definition might be just as useful as Definition 9.1 above.

Definition 10.1 (The weak Albanese of a C-pair)**.** *Let (X, D) be a C-pair where X is compact. A weak Albanese of (X, D) is a normal analytic variety Z° and a morphism*

$$\operatorname{walb}(X, D)^\circ : X^\circ \to Z^\circ$$

such that the following universal property holds. If (S, Δ_S) is any C-semitoric pair and if $s^\circ : (X^\circ, D^\circ) \to (S^\circ, \Delta^\circ_S)$ is any quasi-algebraic C-morphism, then there exists a C-semitoric pair (A, Δ_A), $a \in A^\circ$ and a commutative diagram of morphisms between analytic varieties

$$\begin{tikzcd}
X^\circ \arrow[r, "\operatorname{walb}(X,D)^\circ"'] \arrow[rrr, bend left, "s^\circ"] & Z^\circ \arrow[r, "\iota"'] & A^\circ \arrow[r, "c"'] & S^\circ ,
\end{tikzcd}$$

such that the following holds.

(10.1.1) The morphism ι is a generically injective.

(10.1.2) The composed morphism $\iota \circ \operatorname{walb}(X,D)^\circ$ *is a quasi-algebraic* $\mathcal{C}$*-morphism between the pairs* (X°, D°) *and* $(A^\circ, \Delta_A^\circ)$.

(10.1.3) The morphism c *is a quasi-algebraic* $\mathcal{C}$*-morphism between the pairs* $(A^\circ, \Delta_A^\circ)$ *and* $(S^\circ, \Delta_S^\circ)$.

Comparing Definitions 9.1 and 10.1, one sees that Definition 9.1 does not define the Albanese as a $\mathcal{C}$-semitoric pair. Instead, it defines Z° as the normalized image of a hypothetical Albanese variety and leaves the precise embedding of Z° into a $\mathcal{C}$-semitoric pair unspecified. In this sense, Definition 9.1 does not define the Albanese map. It defines its image.

Conjecture 10.2 (The weak Albanese of a $\mathcal{C}$-pair). Let (X,D) be a $\mathcal{C}$-pair where X is compact Kähler. Then, a weak Albanese of (X,D) exists. The variety Z° and the morphism $\operatorname{walb}(X,D)^\circ$ are unique up to unique isomorphism. The group $\operatorname{Aut}_{\mathcal{O}}(X)$ acts on Z° in a way that makes the morphism $\operatorname{walb}(X,D)^\circ$ equivariant.

With the tools available, we believe that Conjecture 10.2 can be shown without much trouble, but we fear that a full proof would either add another ten pages to this already long paper or needs to be integrated into the proof of Theorem 9.2, extending that proof further and rendering it potentially unfit for human consumption. We will therefore restrict ourselves to the short sketch below and publish details elsewhere.

Sketch of proof. The proof of Theorem 9.2 carries over in large parts. The main difference is Claim 9.13, which asserts that the quotient varieties B_δ° stabilize as soon as we pass to sufficiently fine covers, that is, to covers that factorize via γ. In absence of the hypothesis $q^+_{\mathrm{Alb}}(X,D) < \infty$ this cannot possibly hold true. Instead, we claim that it is not the B_δ° that stabilize, but it is the image sets $\operatorname{img} a_\delta^\circ \subseteq B_\delta^\circ$ that does.

In order to make this precise, one needs to modify Choice 9.10. At the end of Step 1, consider the numbers

$$\begin{aligned} d_\delta &:= \#\text{dimension of } \overline{\operatorname{img} a_\delta^\circ} \subseteq B_\delta, \\ d_{\min} &:= \min\{d_\delta \;:\; \delta \in M\} \\ n_\delta &:= \#\text{components in the typical non-empty fibre of } a_\delta^\circ : X^\circ \to B_\delta^\circ, \\ n_{\min} &:= \min\{n_\delta \;:\; \delta \in M \text{ and } d_\delta = d_{\min}\} \end{aligned}$$

and choose one particular cover $\gamma \in M$ such that $d_\gamma = d_{\min}$ and $n_\gamma = n_{\min}$. Once the choice is made, take

$$Z^\circ := \text{normalization of } \overline{\operatorname{img} a_\gamma^\circ}$$

and let $\operatorname{walb}(X,D)^\circ : X^\circ \to Z^\circ$ be the morphism obtained by applying the universal property of the normalization map to the restricted morphism $a_\delta^\circ : X^\circ \to Z^\circ$. With the appropriate modifications, the required universal properties of $\operatorname{walb}(X,D)^\circ$ will follow by arguments analogous to those in Step 4 of the proof of Theorem 9.2. □

10.2. **Inequalities between augmented irregularities.** If (X,D) is a $\mathcal{C}$-pair where X is compact Kähler, then (5.6.1) immediately gives an inequality between the augmented irregularities

$$q^+_{\mathrm{Alb}}(X,D) \le q^+(X,D). \tag{10.3.1}$$

We do not understand the meaning of Inequality (10.3.1). We do not know if (10.3.1) can ever be strict. If it is strict, this means that adapted differentials come in two types.

- A subset of the adapted differentials comes from the Albanese morphism, at least after passing to suitable high covers.
- The general adapted differential is not induced by any morphism to a $\mathcal{C}$-semitoric pair.

We do not understand this distinction and wonder if there is a geometric explanation, perhaps in Hodge-theoretic terms.

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