C-PAIRS AND THEIR MORPHISMS

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Abstract. This paper surveys Campana's theory of C-pairs (or "geometric orbifolds") in the complex-analytic setting, to serve as a reference for future work. Written with a view towards applications in hyperbolicity, rational points, and entire curves, it introduces the fundamental definitions of C-pair-theory systematically. In particular, it establishes an appropriate notion of "morphism", which agrees with notions from the literature in the smooth case, but it is better behaved in the singular setting and has functorial properties that relate it to minimal model theory.

Contents

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1. Introduction

1.1. **$\mathcal{C}$-pairs.** Following earlier work of Miyaoka and others, Campana introduced "$\mathcal{C}$-pairs" or "geometric orbifolds" to complex-analytic geometry in his influential paper [Cam04]. In simplest terms, $\mathcal{C}$-pairs (X, D) consist of a normal variety X and a $\mathbb{Q}$-divisor with "standard coefficients",

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i, \quad \text{where all } m_i \in \mathbb{N}^{\geq 2} \cup \{\infty\}.$$

Conceptually, $\mathcal{C}$-pairs and their associated differentials interpolate between two "extreme" geometries.

- The geometry of X, which is governed by the sheaves Ω^p_X of Kähler differentials.
- The geometry of $X \setminus D$, which is governed by the sheaves $\Omega^p_X(\log D)$ of logarithmic differentials.

$\mathcal{C}$-pairs appear in higher-dimensional birational geometry, where geometers simplify line bundles by passing to branched covers and use boundary divisors D to keep track of ramification orders. They appear in the study of entire curves and rational points over function fields, where the boundary divisors D encode tangency conditions that are different from (and in some settings more natural than) the conditions imposed by root stacks. In moduli theory, geometers study fibrations and use boundary divisors D as bookkeeping devices for multiplicities of fibres.

Special pairs and pairs of general type. Campana has seen that $\mathcal{C}$-pairs are the natural objects that generalize the dichotomy between rational/elliptic and higher-genus curves to higher dimensions. He attaches to every compact Kähler manifold X a natural "core fibration" that decomposes X into the *fibre space base*, which is a $\mathcal{C}$-pair of general type, and the *fibres*, which are "special" in the sense that they do not admit dominant morphisms to $\mathcal{C}$-pairs of general type. $\mathcal{C}$-pairs of general type and special $\mathcal{C}$-pairs differ in almost every aspect of topology or arithmetic/metric/analytic geometry. One aspect is illustrated by the following conjecture.

Conjecture 1.1 (Campana). —

- Let X be a projective manifold defined over a number field k. Then, X is special if and only if its rational points are potentially dense[1].
- Let X be a complex projective manifold. Then, X is special if and only if X admits a Zariski dense entire curve.

Conjecture 1.1 is a vast generalization of Lang's famous conjectures. It motivates the present paper and follow-up work on the "Albanese of a $\mathcal{C}$-pair" that is currently in preparation.

1.2. **Content of the paper.** This paper surveys Campana's theory of $\mathcal{C}$-pairs (or "geometric orbifolds") in the complex-analytic setting. Aiming to serve as a reference for future work, it introduces the fundamental definitions of $\mathcal{C}$-pairs-theory systematically.

Part I: Adapted tensors and differentials. Following ideas of [Miy08] and [CP15], the first part of the present paper equips $\mathcal{C}$-pairs with a (co)tangent bundle, with sheaves of differential forms, and sheaves of higher-order tensors. These objects have been used in [Miy08] to establish novel Chern class inequalities, and in [CP15] to construct natural foliations on base spaces of families of canonically polarized manifolds.

[1] potentially dense = there exists a finite field extension $k \subseteq k'$ such that k'-rational points are Zariski dense in X.

Conceptually, the "sheaves $\Omega^p_{(X,D)}$ of $\mathcal{C}$-pair-differentials" should interpolate between the sheaves of Kähler– and logarithmic differentials,

$$\Omega^p_X \subseteq \Omega^p_{(X,D)} \subseteq \Omega^p_X(\log D),$$

in the sense that sections of $\Omega^p_{(X,D)}$ should be differentials on X, with logarithmic poles of fractional pole order $\frac{m_i-1}{m_i}$ along the component D_i of D. While "differentials with fractional pole order" do not exist on X in any meaningful way, they can be defined on suitable covers of the space X. Sections 3 and 4 make these vague concepts precise. In order to obtain a theory with good universal properties, these sections define "sheaves of adapted reflexive differentials" on arbitrary covers of arbitrary $\mathcal{C}$-pairs, including very singular ones.

Section 5 shows that adapted reflexive differentials over $\mathcal{C}$-pairs with mild singularities have optimal pull-back properties, similar to the pull-back properties of reflexive differentials on spaces with rational singularities, as established in [GKKP11, Keb13, KS21]. These results relate $\mathcal{C}$-pairs to minimal model theory and will be instrumental when we define morphisms of $\mathcal{C}$-pairs in the second part of this paper.

With all preparations at hand, Section 6 discusses $\mathcal{C}$-pair analogues of several classic invariants, including the "irregularity of $\mathcal{C}$-pair" and a notion of "$\mathcal{C}$-Kodaira-Iitaka dimension for rank-one sheaves of adapted tensors". The section extends the classical vanishing theorem of Bogomolov-Sommese to $\mathcal{C}$-pairs and defines the notion of "special $\mathcal{C}$-pairs".

Part II: Morphisms of $\mathcal{C}$-pairs. The second part of this paper defines and discusses "morphisms of $\mathcal{C}$-pairs". The basic idea is simple: If (X, D_X) and (Y, D_Y) are $\mathcal{C}$-pairs and if $\varphi : X \to Y$ is a morphism of analytic varieties, call φ a morphism of $\mathcal{C}$-pairs if adapted reflexive differentials on Y pull back to adapted reflexive differentials on X. Sections 7 and 8 make this idea precise. Section 9 establishes criteria to guarantee that a given morphism of varieties is a morphism of $\mathcal{C}$-pairs. To illustrate these concepts and highlight some features of our definition, Section 10 discusses several (non-)examples. Sections 11, 12 and 13 establish functoriality properties, the existence of categorical quotients, and relate morphisms of $\mathcal{C}$-pairs to basic notions of minimal model theory.

Section 14 compares our notion "morphism" with other notions that have appeared in the literature. Among those, Campana's definition of an *orbifold morphisms* is perhaps the most prominent: if (X, D_X) and (Y, D_Y) are $\mathcal{C}$-pairs and if $\varphi : X \to Y$ is holomorphic, then φ is called orbifold morphism if a purely numerical criterion holds, relating the coefficients of boundary divisors D_X and D_Y with multiplicities of pull-back divisors coming from Y. To ensure that all quantities are well-defined, Campana defines orbifold morphisms only in settings where

(1.2.1) the space Y is $\mathbb{Q}$-factorial, and
(1.2.2) the morphism φ does not take its image inside the support of D_Y.

For morphism between smooth pairs that satisfy (1.2.2), Campana's definition coincides with ours, and is generally easier to check. For singular pairs, Campana's definition and ours do not coincide in general, even in cases where all pairs are uniformizable. The difference will be of great importance for the applications to Conjecture 1.1, where singularities naturally appear through minimal model theory and purely numerical data might not always suffice to capture the conceptually right geometric picture.

Section 15 gathers several open questions and mentions problems for future research.

1.3. **Outlook.** This publication is the first in a series; two follow-up papers will likely appear later this year. Building on notions and fundamental results obtained here, a second paper, [KR24a], will introduce "$\mathcal{C}$-semitoric varieties" as analogues of Abelian varieties and (semi)tori used in the classic complex geometry. It follows Serre by defining the

Albanese of a C-pair as the universal map to a C-semitoric variety and shows that the Albanese exists in relevant cases.

Aiming to establish hyperbolicity properties of C-pairs with large irregularity, a third paper, [KR24b], will extend the fundamental theorem of Bloch-Ochiai to the context of C-pairs. Building on works of Kawamata, Ueno, and Noguchi, it recasts parabolic Nevanlinna theory as a “Nevanlinna theory for C-pairs”. The authors hope that this approach might be of independent interest

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2. Notation and standard facts

This paper works in the category of complex analytic spaces and is written with a view towards a future *C-Nevanlinna theory*. All the material in this paper will however work in the complex-algebraic setting, typically with simpler definitions and proofs. We expect that large parts of this paper will work for algebraic varieties over perfect fields of arbitrary characteristic and refer the reader to [KPS22], which discusses C-curves over global function fields.

2.1. **Global conventions.** With very few exceptions, we follow the notation of the standard reference texts [GR84, Dem12, NW14]. An *analytic variety* is a reduced, irreducible complex space. For clarity, we refer to holomorphic maps between analytic varieties as *morphisms* and reserve the word *map* for meromorphic mappings.

Definition 2.1 (Big and small sets)**.** *Let X be an analytic variety. An analytic subset $A \subsetneq X$ is called* small *if it has codimension two or more. An open set $U \subseteq X$ is called* big *if $X \setminus U$ is analytic and small.*

2.2. **Sheaves.** The following notation will be used to speak about “meromorphic differentials” on normal spaces. We refer the reader to [Sta21, Tag 01X1] for a further discussion of meromorphic sections of coherent sheaves.

Notation 2.2 (Meromorphic functions). If X is any normal analytic variety, write $\mathscr{M}_X$ for the sheaf of meromorphic functions. If $\mathscr{E}$ is any reflexive, coherent sheaf on X, we call $\mathscr{E} \otimes \mathscr{M}_X$ the *sheaf of meromorphic sections in $\mathscr{E}$*.

Notation 2.3 (Meromorphic section with prescribed location of poles). Let X be a normal analytic variety and let $D \in \operatorname{Div}(X)$ be a Weil divisor. If $\mathscr{E}$ is any coherent, locally free sheaf on X, write $\mathscr{E}(*D) \subset \mathscr{E} \otimes \mathscr{M}_X$ for the sheaf of meromorphic sections in $\mathscr{E}$ that are allowed to have poles along the support of D.

Notation 2.4 (Differentials with logarithmic poles). Let X be a normal analytic variety and let $D \in \mathbb{Q}\operatorname{Div}(X)$ be an effective Weil $\mathbb{Q}$-divisor on X, with nc support. For brevity, we will often write $\Omega^p_X(\log D)$ to denote the sheaves of Kähler differentials with logarithmic poles along $\operatorname{supp} D$. We will use the correct, but longer forms $\Omega^p_X(\log \operatorname{supp} D)$ or $\Omega^p_X(\log D_{\mathrm{red}})$ only if confusion is likely to arise.

This paper frequently works with reflexive sheaves on normal analytic varieties. The following notation will be used.

Notation 2.5 (Reflexive hull and reflexive pull-back). If $\mathscr{E}$ is any coherent sheaf on a normal analytic variety X, we will frequently write $\mathscr{E}^{\vee\vee}$ for its double dual, which is a coherent, reflexive sheaf. We refer to $\mathscr{E}^{\vee\vee}$ as the *reflexive hull* of $\mathscr{E}$. If $n \in \mathbb{N}$ is any number, define the *reflexive tensor power* and *reflexive symmetric power* of $\mathscr{E}$ as

$$\mathscr{E}^{[\otimes n]} := \left(\mathscr{E}^{\otimes n}\right)^{\vee\vee} \quad \text{and} \quad \operatorname{Sym}^{[n]} \mathscr{E} := \left(\operatorname{Sym}^n \mathscr{E}\right)^{\vee\vee}.$$

Given any Weil divisor $D \in \operatorname{Div}(X)$, define the *reflexive twist of $\mathscr{E}$ by D* as

$$\mathscr{E}(D) := \left(\mathscr{E} \otimes \mathscr{O}_X(D)\right)^{\vee\vee}.$$

If $\varphi : Y \to X$ is any morphism from a normal analytic variety Y, we will often write $\varphi^{[*]}\mathscr{E} := (\varphi^*\mathscr{E})^{\vee\vee}$ and refer to this sheaf as the *reflexive pull-back* of $\mathscr{E}$.

Notation 2.6 (Reflexive differentials). Let X be a normal analytic variety and write $\Omega_X^{[p]} := \left(\Omega_X^p\right)^{\vee\vee}$. Given a Weil $\mathbb{Q}$-divisor $D \in \mathbb{Q}\operatorname{Div}(X)$, write $\Omega_X^{[p]}(\log D) := \left(\Omega_X^p(\log D)\right)^{\vee\vee}$. By minor abuse of language, we refer to sections in these sheaves as *reflexive (logarithmic) differentials*.

Remark 2.7 (Reflexive differentials and prolongations). In the setting of Notation 2.6, if $X^+ \subset X$ is the maximal open set where $(X, \operatorname{supp} D)$ is nc and if $\iota : X^+ \to X$ denotes the inclusion map, then there exists a canonical isomorphism $\Omega_X^{[p]}(\log D) \cong \iota_*\Omega_{X^+}^p(\log D)$. We refer the reader to [Ser66, Thm. 1 and Prop. 7] for details.

Reminder 2.8 (Reflexive sheaves of rank one, base loci and meromorphic maps). If $\mathscr{F}$ is any rank-one, coherent reflexive sheaf on a normal analytic variety X, then $\mathscr{F}|_{X_{\mathrm{reg}}}$ is locally free. If $V \subseteq H^0\left(X, \mathscr{F}\right)$ is non-trivial and finite-dimensional, then the associated holomorphic

$$X_{\mathrm{reg}} \setminus (\text{Base locus } V) \to \mathbb{P}(V^\vee)$$

extends to a meromorphic mapping $X \dashrightarrow \mathbb{P}(V^\vee)$. For a proof, recall a fundamental result of Rossi, [Ros68, Thm. 3.5] or [GR70, Thm. 1.1]: there exists a proper modification $\pi : \widetilde{X} \twoheadrightarrow X$ such that $\pi^*\mathscr{F}/\text{tor}$ is locally free, hence invertible.

2.3. **Weil $\mathbb{Q}$-divisors and pairs.** Much of our discussion is centred around Weil $\mathbb{Q}$-divisors on normal analytic varieties. The following standard language will be used.

Notation 2.9 ($\mathbb{Q}$-Cartier and locally $\mathbb{Q}$-Cartier divisors). Let X be a normal analytic variety and let $D \in \mathbb{Q}\operatorname{Div}(X)$ be a Weil $\mathbb{Q}$-divisor on X.

(2.9.1) The divisor D is called *$\mathbb{Q}$-Cartier* if there exists a positive number $m \in \mathbb{N}^+$ such that $m \cdot D$ is integral and Cartier.

(2.9.2) The divisor D is called *locally $\mathbb{Q}$-Cartier* if every point $x \in X$ has an open neighbourhood $U = U(x) \subseteq X$ such that $D|_U$ is $\mathbb{Q}$-Cartier on U.

Notation 2.10 ($\mathbb{Q}$-factorial and locally $\mathbb{Q}$-Cartier varieties). Let X be a normal analytic variety.

(2.10.1) The variety X is called *$\mathbb{Q}$-factorial* if every Weil $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

(2.10.2) The variety X is called *locally $\mathbb{Q}$-factorial* if there exists a basis of topology, $(U_\alpha)_{\alpha\in A}$, where all $U_\bullet$ are $\mathbb{Q}$-factorial analytic varieties.

Remark 2.11 (Local and global properties in analytic geometry). In contrast to algebraic geometry, the local properties in Notation 2.9 and 2.10 do not imply the global ones. It is not hard to construct a non-compact, normal analytic surface S and a prime Weil divisor $D \in \operatorname{Div}(S)$ with the following properties.

(2.11.1) The singular locus $S_{\mathrm{sing}} = \{s_1, s_2, \ldots\}$ is countable infinite, discrete, and contained in the support of D. In particular, S has only isolated singularities.

(2.11.2) If $n \in \mathbb{N}^+$ is any number, then $n \cdot D$ is Cartier on the set $S_{\mathrm{reg}} \cup \{s_n\}$, which is an open neighbourhood of the singular point s_n. If $m \in \mathbb{N}^+$ is larger than n, then $n \cdot D$ is *not* Cartier in any neighbourhood of the singular point s_m.

In these examples, the divisor D is locally $\mathbb{Q}$-Cartier but not $\mathbb{Q}$-Cartier. Similar examples exist for factoriality.

Notation 2.12 (Operations on Weil $\mathbb{Q}$-divisors). Let X be a normal analytic variety and let $D \in \mathbb{Q}\,\mathrm{Div}(X)$ be a Weil $\mathbb{Q}$-divisor on X. Following standard notation, we denote the *round-down*, *round-up* and the *fractional part* of D by $\lfloor D \rfloor$, $\lceil D \rceil$, and $\{D\} := D - \lfloor D \rfloor$, respectively. We call D *reduced* if all its coefficients are one, and write D_{red} for the reduced Weil divisor obtained by setting all non-vanishing coefficients of D to one.

Definition 2.13 (Pair, log pair, nc pair)**.** *A* pair *is a tuple* (X, D) *consisting of a normal analytic variety* X *and a Weil* $\mathbb{Q}$*-Divisor* D *on* X *with coefficients in the interval* $(0 \dots 1] \cap \mathbb{Q}$.

- *Call* (X, D) *a* log pair *if* D *is reduced.*
- *Call* (X, D) *a* nc pair *if* X *is smooth and* D *has normal crossing support.*

Notation 2.14 (Open part of a pair). Let (X, D) be a pair. The *open part* is the pair (X°, D°), where $X^\circ := X \setminus \operatorname{supp}\lfloor D \rfloor$ and $D^\circ := D \cap X^\circ$.

Observation 2.15 (Open sets where pair is nc). Let (X, D) be a pair. Since the property "nc pair" is local and open, there exists a maximal open subset $X^+ \subseteq X$ where the pair is nc. Observe that this subset is big.

Definition 2.16 (Gorenstein conditions)**.** —

- *Call a pair* (X, D) Gorenstein *if the* $\mathbb{Q}$*-divisor* D *is integral and the sheaf*
$$\big(\omega_X \otimes \mathscr{O}_X(D)\big)^{\vee\vee}$$
is locally free.
- *Call a pair* (X, D) $\mathbb{Q}$-Gorenstein *if there exists a number* $m \in \mathbb{N}^+$ *such that the divisor* $m \cdot D$ *is integral and the sheaf*
$$\big(\omega_X^{\otimes m} \otimes \mathscr{O}_X(m \cdot D)\big)^{\vee\vee}$$
is locally free.
- *Call a pair* (X, D) locally $\mathbb{Q}$-Gorenstein *if there exists an open covering* $X = \cup_i X_i$ *such that the pairs* $(X_i, D \cap X_i)$ *are* $\mathbb{Q}$*-Gorenstein.*

Remark 2.17 (Gorenstein conditions and the canonical divisor). For pairs (X, D) where a canonical divisor K_X exists[2], the Gorenstein conditions of Definition 2.16 can be reformulated as asking that $K_X + D$ is Cartier, $\mathbb{Q}$-Cartier or locally $\mathbb{Q}$-Cartier, respectively.

2.4. **Covers and q-morphisms.**

Large parts of this paper are concerned with quasi-finite morphism between normal varieties of equal dimension. For brevity, the following notation will be used.

Definition 2.18 (q-morphisms, relative automorphisms)**.** *Quasi-finite morphisms between normal analytic varieties of equal dimension are called* q-morphisms*. If* $\gamma : \widehat{X} \to X$ *is a* q*-morphism, consider the* relative automorphism group
$$\operatorname{Aut}_{\mathscr{O}}\big(\widehat{X}/X\big) := \big\{ g \in \operatorname{Aut}_{\mathscr{O}}(X) \;:\; \gamma \circ g = \gamma \big\}.$$

Reminder 2.19 (Openness). Recall that q-morphisms are open, [GR84, Sect. 3.2]. The relative automorphism group of a q-morphism is finite and acts holomorphically.

Reminder 2.20 (Pull-back of Weil $\mathbb{Q}$-divisors). If $\gamma : \widehat{X} \to X$ is a q-morphism, there exists a well-defined pull-back morphism for Weil divisors,
$$\gamma^* : \mathrm{Div}(X) \to \mathrm{Div}(\widehat{X}),$$
that agrees over X_{reg} with the standard pull-back of Cartier divisors, respects linear equivalence and therefore induces a morphism between Weil divisor class groups.

[2] A canonical divisor exists if the sheaf ω_X has a meromorphic section. This is the case if the normal analytic variety X is Stein or quasi-projective. A compact complex manifold of algebraic dimension zero need not have a canonical divisor.

Definition 2.21 (Covers, Galois covers). *Finite morphisms between normal analytic varieties of equal dimension are called* covers. *A cover $\gamma : X \to Y$ is called* Galois *if it is isomorphic to the quotient morphism*

$$q : X \to X/\mathrm{Aut}_{\mathscr{O}}(X/Y).$$

Covers are necessarily surjective. We do *not* require that Galois covers are locally biholomorphic. We refer the reader to [Car57, Thm. 4] for quotients of analytic varieties.

Notation 2.22 (Branch and ramification divisor). If $\gamma : X \twoheadrightarrow Y$ is a cover, write $\mathrm{Branch}(\gamma) \in \mathrm{Div}(Y)$ for the reduced Weil divisor on Y whose support equals the codimension-one part of the branch locus of γ. Analogously, write $\mathrm{Ramification}(\gamma) \in \mathrm{Div}(X)$ for the reduced Weil divisor on X whose support equals the codimension-one part of the ramification locus of γ.

The following lemma allows us to compare logarithmic differentials on the domain and target of a cover. The proof follows most easily from a local computation. We refer the reader to [GKK10, Sect. 2.C] for details and for related results.

Lemma 2.23 (Criterion for log poles). *Let $\gamma : X \twoheadrightarrow Y$ be a cover, let D be a reduced Weil divisor on Y, and let σ be a meromorphic differential form with poles along D. Assume that the pairs (Y, D) and (X, γ^*D) are both nc. Then, the following statements are equivalent.*

(2.23.1) The form $\sigma \in H^0\big(Y,\ \Omega^p_Y(*D)\big)$ has logarithmic poles.

(2.23.2) The pull-back form $(d\,\gamma)\sigma \in H^0\big(X,\ \Omega^p_X(*\gamma^*D)\big)$ has logarithmic poles. □

2.5. **$\mathcal{C}$-pairs and adapted morphisms.** The key notion of the present paper is the $\mathcal{C}$-pair, also called *orbifolde géométrique* by Campana [Cam11, Def. 2.1] or *Campana constellation* by Abramovich [Abr09, Lecture 2]. We recall the definition for the reader's convenience.

Definition 2.24 ($\mathcal{C}$-pairs, see [Cam11, Sect. 2.1]). *A $\mathcal{C}$-pair is a pair (X, D) where the Weil $\mathbb{Q}$-divisor D is of the form*

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i,$$

with $m_i \in \mathbb{N}^{\geq 2} \cup \{\infty\}$ and $\frac{\infty - 1}{\infty} = 1$.

Notation 2.25 ($\mathcal{C}$-multiplicity). In the setting of Definition 2.24, if $H \subset X$ is any prime divisor, define the *$\mathcal{C}$-multiplicity of D along H* as

$$\mathrm{mult}_{\mathcal{C},H} D := \begin{cases} m_i & \text{if there exists an index } i \text{ with } H = D_i \\ 1 & \text{otherwise.} \end{cases}$$

It will sometimes be convenient to consider the following Weil $\mathbb{Q}$-divisor

$$D_{\mathrm{orb}} := \sum_{i \,|\, m_i < \infty} \frac{1}{m_i} \cdot D_i \in \mathbb{Q}\,\mathrm{Div}(X).$$

$\mathcal{C}$-pairs come with a class of distinguished morphisms, called *adapted morphisms*.

Definition 2.26 (Adapted morphism, compare [CP19, Sect. 5.1]). *Consider a $\mathcal{C}$-pair (X, D). A q-morphism $\gamma : \widehat{X} \to X$ is called* adapted *for (X, D) if $\gamma^* D_{\mathrm{orb}}$ is integral. The morphism γ is called* strongly adapted *for (X, D) if $\gamma^* D_{\mathrm{orb}}$ is reduced.*

The word "adapted" is not used uniformly in the literature. Morphisms that we call "adapted" are called "subadapted" in [JK11] and other papers.

Observation 2.27 (Composition). In the setup of Definition 2.26, let

$$\widehat{X}_1 \xrightarrow{\gamma_1} \widehat{X}_2 \xrightarrow{\gamma_2} X$$

be a sequence of q-morphisms. If γ_2 is adapted for (X, D), then so is $\gamma_2 \circ \gamma_1$. □

2.5.1. *Uniformization.* The following Sections 3 and 4 introduce adapted (reflexive) differentials, a class of differential forms that exist on the domain of a q-morphism. Uniformizations are adapted morphisms where these differentials take a particularly simple form. Example 4.6 on page 21 compares adapted differentials with Kähler differentials and makes this vague statement precise.

Definition 2.28 (Uniformization of a C-pair)**.** *Let (X, D) be a C-pair. A* uniformization *of (X, D) is a strongly adapted cover $u : X_u \twoheadrightarrow X$ where $\operatorname{Branch}(u) \subseteq \operatorname{supp} D$ and $(X_u, \operatorname{supp} u^*\lfloor D \rfloor)$ has normal crossings.*

Example 2.29 (Neil's parabola and three lines through a common point). The C-pairs

$$\left(\mathbb{C}^2,\ \tfrac{1}{2}\cdot\{x^2 = y^3\}\right) \quad \text{and} \quad \left(\mathbb{C}^2,\ \tfrac{2}{3}\cdot\{x=0\} + \tfrac{2}{3}\cdot\{y=0\} + \tfrac{1}{2}\cdot\{x=y\}\right)$$

are uniformizable. The proof is part of the classification of orbifaces with smooth base, [Ulu07]. In each example, one computes that the orbifold fundamental group is finite. The associated orbifold-universal branched covering is a locally simply connected, normal surface and hence smooth by Mumford's classic result [Mum61, Thm. on p. 229]. We refer the reader to [Ulu07, Sect. 3 and Thm. 3.2] for precise statements, a full classification with more examples, and details.

Remark 2.30 (Branch divisor and branch locus). Recall from Notation 2.22 that $\operatorname{Branch}(u)$ denotes the codimension-one part of the branch locus for the morphism u. Uniformizations may branch over a codimension-two set that is not contained in $\operatorname{supp} D$.

Remark 2.31 (Uniformizations and q-morphisms). Let (X, D) be a C-pair and let $\gamma : \widehat{X} \to X$ be any q-morphism. Then, there exists a maximal open subset $X^+ \subseteq X$ over which γ is a uniformization. The set $X^+ \subseteq X$ is Zariski open, but not necessarily big.

Definition 2.32 (Uniformizable C-pairs)**.** *A C-pair (X, D) is* uniformizable *if there exists a uniformization. It is* locally uniformizable *if every point of X has a uniformizable neighbourhood.*

Remark 2.33. If a C-pair is nc, then it is locally uniformizable. If (X, D) is any C-pair, then there exists a maximal open subset $X^{\mathrm{lu}} \subseteq X$ over which (X, D) is locally uniformizable. The set $X^{\mathrm{lu}} \subseteq X$ is Zariski open and big.

Remark 2.34 ($\mathbb{Q}$-factoriality of (locally) uniformizable pairs). Uniformizable pairs are $\mathbb{Q}$-factorial. Locally uniformizable pairs are locally $\mathbb{Q}$-factorial. Both statements follow [KM98, Lem. 5.16], whose (short) proof applies without change in the analytic setting.

Remark 2.35 (Singularities of locally uniformizable pairs). If a C-pair (X, D) is locally uniformizable, then the pair (X, D) is klt, [KM98, Prop. 5.13]. Recalling that klt singularities are rational, [KM98, Thm. 5.22] and [Fuj22, Thm. 3.12][3], it follows that X has rational singularities.

2.5.2. *Existence of adapted covers.* A standard computation shows that strongly adapted covers always exist locally.

Lemma 2.36 (Strongly adapted covers exist locally)**.** *Let (X, D) be a C-pair as in Definition 2.24. Assume that one of the following holds.*

(2.36.1) The space X is Stein and the divisor D has only finitely many components.
(2.36.2) The space X is projective.

If $x \in X \setminus \operatorname{supp}\{D\}$ is any point, then there exists a strongly adapted Galois cover with cyclic group that is locally biholomorphic over x.

[3] See also the vanishing theorems proven in [Fuj23].

Proof. We consider only the Stein setting and refer the reader to [Laz04, Sect. 4.1.B] for the projective setting. The assumption that X is Stein guarantees that there exist functions $f_i \in \mathscr{O}_X(X)$ such that the following holds for every index i.

- The function f_i vanishes along the Weil divisor D_i to order one.
- The function f_i does not vanish along any of the Weil divisors D_j, for $j \neq i$.
- The function f_i does not vanish at x.

We can then set $n := \operatorname{lcm}\{m_i \,:\, m_i < \infty\}$ and

$$\widehat{X} := \text{normalisation of } \Big\{(x, y) \in X \times \mathbb{A}^1 \,:\, y^n = \prod_{i \,|\, m_i < \infty} f_i^{n/m_i}(x)\Big\}. \qquad \square$$

Remark 2.37. In the proof of Lemma 2.36, if the components of $\{D\}$ are Cartier and linearly trivial, then we can find functions f_i that vanish only at D_i, and the construction will yield a cover that is locally biholomorphic away from $\operatorname{supp}\{D\}$.

Remark 2.38. If (X, D) is a C-pair where X is compact but not projective, then it is not clear that an adapted cover exists.

2.5.3. *C-pairs and root stacks.* Looking at the definition of an adapted morphism, the reader might wonder about the relation of C-pairs and root stacks. We argue that the two notions are conceptually quite different. For now, observe that Definitions 2.24 and 2.26 do not ask that D or D_i are Cartier. For the construction of root stack, the Cartier assumption is however essential.

2.6. **Weil divisorial sheaves.** Integral Weil divisors on normal spaces define *Weil divisorial sheaves*, that is, coherent, reflexive sheaves of rank one. Since we will later need to discuss Weil divisorial sheaves on singular spaces, we briefly recall the relevant definitions, facts and constructions.

Notation 2.39 (Weil divisorial sheaves). Let X be a normal analytic variety and let $D = \sum m_i \cdot D_i$ be an effective Weil divisor on X. Following standard notation, consider the associated Weil divisorial sheaf

$$\mathscr{O}_X(-D) = \Big(\bigotimes_i \mathscr{J}_{D_i}^{\otimes m_i}\Big)^{\vee\vee}.$$

By construction, this sheaf is reflexive of rank one, and comes with a natural embedding $\mathscr{O}_X(-D) \hookrightarrow \mathscr{O}_X$. We denote the quotient by $\mathscr{O}_D := \mathscr{O}_X / \mathscr{O}_X(-D)$.

Remark 2.40 (Inclusions and projections). Let X be a normal analytic variety and let $D_1 \leq D_2$ be two effective Weil divisors on X. Then, $\mathscr{O}_X(-D_2) \subseteq \mathscr{O}_X(-D_1)$, and so there exists a natural surjection $\mathscr{O}_{D_2} \twoheadrightarrow \mathscr{O}_{D_1}$.

In the setting of Reminder 2.20, where a meaningful pull-back of a Weil divisor can be defined, the sheaf $\mathscr{O}_{\widehat{X}}(-\gamma^* D)$ associated to a pull-back divisor is generally not equal to the pull-back sheaf $\gamma^* \mathscr{O}_X(-D)$; note that the pull-back sheaf need not be reflexive or torsion free. Still, there exists a comparison morphism that becomes isomorphic if D is Cartier.

Remark 2.41 (Pull-back of divisors and sheaves). Let $\gamma : \widehat{X} \to X$ be a q-morphism and let $D \in \operatorname{Div}(X)$ be an effective Weil divisor on X. By construction, there exists a canonical morphism

$$\gamma^* \mathscr{O}_X(-D) \to \mathscr{O}_{\widehat{X}}\big(-\gamma^* D\big)$$

that is isomorphic over the big open subset of X where D is Cartier. If D is Cartier on all of X, then $\mathscr{O}_X(-D)$ is invertible, the pull-back sequence reads

$$0 \to \mathscr{O}_{\widehat{X}}(-\gamma^* D) \to \mathscr{O}_{\widehat{X}} \to \gamma^* \mathscr{O}_D \to 0,$$

and is exact. It follows that $\gamma^* \mathscr{O}_D = \mathscr{O}_{\gamma^* D}$.

$$\begin{array}{ccccc} \Omega^1_X & \subseteq & \Omega^1_{(X,D)} & \subseteq & \Omega^1_X(\log D) \\ \langle dz \rangle_{\mathscr{O}_X} & \subseteq & \langle z^{-\frac{m-1}{m}} \cdot dz \rangle_{\mathscr{O}_X} & \subseteq & \langle z^{-1} \cdot dz \rangle_{\mathscr{O}_X}. \end{array}$$

TABLE 1. Hypothetical sheaves and generators on $X = \mathbb{A}^1$

3. Adapted tensors

If X is a compact Kähler manifold, the sheaves Ω^p_X of holomorphic differentials govern the geometry and topology of X. If $D \in \operatorname{Div}(X)$ is a smooth prime divisor, the sheaves $\Omega^p_X(\log D)$ of logarithmic differentials govern the geometry and topology of $X \setminus D$.

The sequence $(X, \frac{n-1}{n} \cdot D)$ of $\mathcal{C}$-pairs is meant to interpolate between the compact manifold X and the non-compact manifold $X \setminus D$. Along these lines, the *sheaves of adapted differentials* are meant to interpolate between the sheaves Ω^p_X and the larger sheaf $\Omega^1_X(\log D)$. Conceptually, an adapted differential is a differential on X with a logarithmic pole of fractional pole order $\frac{n-1}{n}$ along D. Differentials with fractional pole order do not exist on X in any meaningful way. To make the vague concept precise, we work on covers, where adapted differentials can be defined as differentials that "locally look" as if they were the pull-back of differentials on X that had poles of the appropriate order.

Relation to the literature. Adapted differentials have been discussed in the literature, but usually not in a very systematic fashion. We refer the reader to Campana's original papers [Cam04, Cam11], to the paper [CP19] of Campana and Păun, and to Miyaoka's classic [Miy08] for details, applications and further references. The paper [KPS22, Sect. 6] discusses related definitions in positive characteristic, the paper [CKT21, Sect. 3] spells out equivalent, but perhaps more elementary-looking definitions.

Taking a somewhat different point of view, Pedro Núñez considered the algebraic setting, where X is a scheme, and constructed a presheaf of "adapted differentials" on the category Sch/X that is a sheaf with respect to the qfh-topology, [Nú23a]. If everything is algebraic, the constructions outlined in this section are compatible with those of Núñez.

3.1. **Sample computations.** To prepare the reader for the somewhat technical discussion in Definition 3.2 on page 12, we illustrate the main ideas in two simple cases first. The reader who is already familiar with "adapted differentials" might want to skip this section.

3.1.1. *Sample computation in dimension one.* Equip $\mathbb{A}^1$ with a coordinate z and consider the simple case where

$$(X, D) = \left(\mathbb{A}^1, \tfrac{m-1}{m} \cdot \{0\}\right), \quad \text{for one number } m \in \mathbb{N}^+.$$

The sheaf $\Omega^1_{(X,D)}$ of differential forms with logarithmic poles of order $\frac{m_i-1}{m_i}$ should then ideally look as shown in Table 1. The reader will immediately note that $z^{-\frac{m-1}{m}}$ cannot possibly exist as a single-valued function on X, unless $m = 1$ or $m = \infty$. To resolve the problem, write $\widehat{X} := \mathbb{A}^1$, choose a number $\alpha \in \mathbb{N}^+$ and consider the cover

$$\gamma : \widehat{X} \to X, \quad z \mapsto z^{\alpha \cdot m}.$$

Applying the formal rules of derivation, we find that

$$d\gamma\left(z^{-\frac{m-1}{m}} \cdot dz\right) = \alpha m \cdot z^{\alpha - 1} \cdot dz.$$

Accordingly, as shown in Table 2, there exists a sheaf $\Omega^1_{(X,D,\gamma)}$ of differentials on $\widehat{X}$ that looks as if it was the pull-back $\gamma^*\Omega^1_{(X,D)}$ of the hypothetical sheaf $\Omega^1_{(X,D)}$ on X. The inclusion $\gamma^*\Omega^1_X \subseteq \Omega^1_{(X,D,\gamma)}$ allows interpreting sections in $\Omega^1_{(X,D,\gamma)}$ as meromorphic sections

$$\begin{array}{ccccc} \gamma^*\Omega^1_X & \subseteq & \Omega^1_{(X,D,\gamma)} & \subseteq & \gamma^*\Omega^1_X(\log D) \\ \big\langle z^{\alpha m-1}\cdot dz\big\rangle_{\mathscr{O}_{\widehat{X}}} & \subseteq & \big\langle z^{\alpha-1}\cdot dz\big\rangle_{\mathscr{O}_{\widehat{X}}} & \subseteq & \big\langle z^{-1}\cdot dz\big\rangle_{\mathscr{O}_{\widehat{X}}}. \end{array}$$

TABLE 2. Sheaves and generators on $\widehat{X} = \mathbb{A}^1$

$$\begin{array}{ccccc} \Omega^1_X & \subseteq & \Omega^1_{(X,D)} & \subseteq & \Omega^1_X(\log D) \\ \big\langle dx,\, dy\big\rangle_{\mathscr{O}_X} & \subseteq & \big\langle dx,\, y^{-\frac{m-1}{m}}\cdot dy\big\rangle_{\mathscr{O}_X} & \subseteq & \big\langle dx,\, y^{-1}\cdot dy\big\rangle_{\mathscr{O}_X}. \end{array}$$

TABLE 3. Hypothetical sheaves and generators on $X = \mathbb{A}^2$

$$\begin{array}{ccccc} \gamma^*\Omega^1_X & \subseteq & \Omega^1_{(X,D,\gamma)} & \subseteq & \gamma^*\Omega^1_X(\log D) \\ \big\langle dx,\, y^{\alpha m-1}\cdot dy\big\rangle_{\mathscr{O}_{\widehat{X}}} & \subseteq & \big\langle dx,\, y^{\alpha-1}\cdot dy\big\rangle_{\mathscr{O}_{\widehat{X}}} & \subseteq & \big\langle dx,\, y^{-1}\cdot dy\big\rangle_{\mathscr{O}_{\widehat{X}}}. \end{array}$$

TABLE 4. Sheaves and generators on $\widehat{X} = \mathbb{A}^2$

of $\gamma^*\Omega^1_X$ with a pole of order $\alpha\cdot(m-1)$ along $\{0\}$. Given that $\gamma^*D = \alpha(m-1)\cdot\{0\}$, we can write $\Omega^1_{(X,D,\gamma)}$ in a coordinate-free way as

$$\Omega^1_{(X,D,\gamma)} = \mathscr{O}_{\widehat{X}}(\gamma^*D)\otimes\gamma^*\Omega^1_X. \tag{3.0.1}$$

3.1.2. *Sample computation in dimension two.* Next, equip $\mathbb{A}^2$ with coordinates x, y and consider the pair

$$(X,D) = \Big(\mathbb{A}^2,\, \tfrac{m-1}{m}\cdot\{y=0\}\Big), \quad \text{for one number } m\in\mathbb{N}^+.$$

The sheaf $\Omega^1_{(X,D)}$ of differential forms with logarithmic poles of order $\frac{m_i-1}{m_i}$ should then ideally look as shown in Table 3. As before, write $\widehat{X} := \mathbb{A}^2$, choose a number $\alpha\in\mathbb{N}^+$ and consider the cover

$$\gamma : \widehat{X}\to X, \quad (x,y)\mapsto\big(x, y^{\alpha\cdot m}\big).$$

Again, we find a sheaf $\Omega^1_{(X,D,\gamma)}$ of differentials on $\widehat{X}$ that looks as if it was the pull-back $\gamma^*\Omega^1_{(X,D)}$ of the hypothetical sheaf $\Omega^1_{(X,D)} = \langle dx, y^{-\frac{m-1}{m}}\cdot dy\rangle_{\mathscr{O}_X}$. Table 4 spells out the details. In contrast to the one-dimension case, observe that Equation (3.0.1) no longer describes $\Omega^1_{(X,D,\gamma)}$ in the present setting. In fact, it turns out that

$$\mathscr{O}_{\widehat{X}}(\gamma^*D)\otimes\gamma^*\Omega^1_X = \big\langle y^{-\alpha\cdot(m-1)}\cdot dx,\, y^{\alpha-1}\cdot dy\big\rangle_{\mathscr{O}_{\widehat{X}}}$$

is a strict supersheaf of $\Omega^1_{(X,D,\gamma)}$. To this end, observe that sections in $\mathscr{O}_{\widehat{X}}(\gamma^*D)\otimes\gamma^*\Omega^1_X$ are required to have the correct pole order, but need not have logarithmic poles. The following coordinate-free description avoids that problem,

$$\Omega^1_{(X,D,\gamma)} = \underbrace{\mathscr{O}_{\widehat{X}}(\gamma^*D)\otimes\gamma^*\Omega^1_X}_{=:\mathscr{A}_{1,1}} \cap \underbrace{\Omega^1_{\widehat{X}}(\log\gamma^*D)}_{=:\mathscr{B}_{1,1}}. \tag{3.0.2}$$

Here, the intersection takes place in the sheaf $\mathscr{O}_{\widehat{X}}(*\gamma^*D)\otimes\big(\gamma^*\Omega^1_X\big)$ that contains both $\mathscr{A}_{1,1}$ and $\mathscr{B}_{1,1}$.

3.1.3. *Further generalizations.* In principle, one would like to take (3.0.2) as the definition of $\Omega^1_{(X,D,\gamma)}$. There are, however, several further generalizations that we would like to consider.

Arbitrary q-morphisms: The morphisms γ that we considered in Section 3.1.1–3.1.2 were adapted covers of the pair (X, D). In practise, adapted covers do not always exist[4]. For technical reasons, we will need to define $\Omega^1_{(X,D,\gamma)}$ for q-morphisms γ that are not necessarily adapted. There, we use the round-down of $\gamma^* D$ as the best approximation, replacing the sheaf $\mathscr{A}_{1,1}$ of (3.0.2) by

$$\mathscr{A}_{1,1} := \mathscr{O}_{\widehat{X}}\bigl(\lfloor \gamma^* D \rfloor\bigr) \otimes \gamma^* \Omega^1_X.$$

Logarithmic boundary components: Sections 3.1.1–3.1.2 considered pairs (X, D) where $\lfloor D \rfloor = 0$. In case where D is reduced, the setting simplifies dramatically, as the "sheaf $\Omega^p_{(X,D)}$ of differential forms with logarithmic poles of order $\frac{\infty - 1}{\infty}$" is simply $\Omega^1_X(\log D)$. In general, where D is allowed to have reduced and non-reduced components, we consider the fractional and integral part of D separately, replacing the sheaf $\mathscr{A}_{1,1}$ of (3.0.2) by

$$\mathscr{A}_{1,1} := \mathscr{O}_{\widehat{X}}\bigl(\lfloor \gamma^* \{D\} \rfloor\bigr) \otimes \gamma^* \Omega^1_X\bigl(\log \lfloor D \rfloor\bigr).$$

Higher-order tensors: In addition to sections of $\Omega^1_\bullet$, we will also need to consider p-forms and more generally sections in symmetric powers of $\Omega^p_\bullet$. Again, this forces us to generalize, replacing $\mathscr{A}_{1,1}$ and $\mathscr{B}_{1,1}$ by the sheaves $\mathscr{A}_{n,p}$ and $\mathscr{B}_{n,p}$ found in Definition 3.2 below.

3.2. **Definition and first examples.** Throughout the present Section 3, we will work in the following setting.

Setting 3.1. Let (X, D) be a $\mathcal{C}$-pair as in Definition 2.24 and let $\gamma : \widehat{X} \to X$ be a q-morphism. Assume that the pairs (X, D) and $\bigl(\widehat{X}, \gamma^* D\bigr)$ are nc.

In view of the sample computations in Section 3.1, we hope that the following definition will not come as a surprise.

Definition 3.2 (Adapted tensors)**.** *Assume Setting 3.1. Given numbers $n, p \in \mathbb{N}^+$, consider the sheaves*

$$\mathscr{A}_{n,p} := \mathscr{O}_{\widehat{X}}\bigl(\lfloor n \cdot \gamma^* \{D\} \rfloor\bigr) \otimes \gamma^* \operatorname{Sym}^n \Omega^p_X\bigl(\log \lfloor D \rfloor\bigr)$$
$$\mathscr{B}_{n,p} := \operatorname{Sym}^n \Omega^p_{\widehat{X}}(\log \gamma^* D).$$

Observe that both are subsheaves of $\mathscr{O}_{\widehat{X}}(\gamma^* D) \otimes \bigl(\gamma^* \operatorname{Sym}^n \Omega^p_X\bigr)$ and define the bundle of adapted (n, p)-tensors *as the intersection*

$$\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,D,\gamma)} := \mathscr{A}_{n,p} \cap \mathscr{B}_{n,p}.$$

Collectively, we refer to $\operatorname{Sym}^\bullet_{\mathcal{C}} \Omega^\bullet_{(X,D,\gamma)}$ *as the* bundles of adapted tensors.

Definition 3.3 (Adapted differentials, cf. [CP19, Sect. 5.2])**.** *Assume Setting 3.1. Given a number $p \in \mathbb{N}^+$, define the* bundle of adapted p-forms *as* $\Omega^p_{(X,D,\gamma)} := \operatorname{Sym}^1_{\mathcal{C}} \Omega^p_{(X,D,\gamma)}$. *Collectively, we refer to* $\Omega^\bullet_{(X,D,\gamma)}$ *as the* bundles of adapted differentials. *The sheaf* $\Omega^1_{(X,D,\gamma)}$ *is called* $\mathcal{C}$-cotangent bundle.

The remainder of Section 3 lists properties of $\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,D,\gamma)}$ that will become relevant later. The proofs are elementary but tedious, and will typically involve a computation in local coordinates, or numerical arguments of the form $\lfloor \gamma^* D \rfloor \geq \gamma^* \lfloor D \rfloor$. To keep the size of this (already long) paper within reasonable limits, we refrain from giving full proofs and

[4] A compact complex manifold of algebraic dimension zero need not have any global covers at all!

formulate a sequence of statements euphemistically called "observations", that is, homework left for the reader. The first observation justifies the word "bundle" in Definitions 3.2 and 3.3.

Observation 3.4 (Local freeness). The sheaves $\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,D,\gamma)}$ of Definition 3.2 are locally free. □

The following examples illustrate the construction, expanding Definition 3.2 in a number of special cases.

Example 3.5 (Special cases). In Setting 3.1, assume that numbers $n, p \in \mathbb{N}^+$ are given.

(3.5.1) If $p = \dim X$, then

$$\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,D,\gamma)} = \big(\gamma^* \omega_X^{\otimes n}\big) \otimes \mathcal{O}_{\widehat{X}}\big(\lfloor n \cdot \gamma^* D \rfloor\big).$$

(3.5.2) If $D = 0$, then $\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,0,\gamma)} = \gamma^* \operatorname{Sym}^n \Omega^p_X$.

(3.5.3) If $\gamma = \mathrm{Id}_X$, then $\Omega^p_{(X,D,\mathrm{Id}_X)} = \Omega^p_X(\log \lfloor D \rfloor)$.

(3.5.4) If γ is strongly adapted and $\operatorname{Branch}(\gamma) \subseteq \operatorname{supp} D$, then

$$\operatorname{Sym}^n_{\mathcal{C}} \Omega^p_{(X,D,\gamma)} = \operatorname{Sym}^n \Omega^p_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big).$$

Example 3.6 (Functions with adapted differential). In the setting of Definition 3.2, assume that γ is a cover. Given a function

$$\widehat{f} \in H^0\Big(\widehat{X},\ \mathcal{O}_{\widehat{X}}(-\operatorname{Ramification} \gamma)\Big), \tag{3.6.1}$$

that vanishes along the ramification divisor[5] of γ, an elementary computation in local coordinates shows that the following statements are equivalent.

(3.6.2) The Kähler differential of $\widehat{f}$ is adapted,

$$d\widehat{f} \in H^0\big(\widehat{X},\ \Omega^1_{(X,D,\gamma)}\big) \subseteq H^0\big(\widehat{X},\ \Omega^1_{\widehat{X}}(\log \gamma^* D)\big).$$

(3.6.3) The zero-divisor of the function $\widehat{f}$ satisfies the inequality

$$\operatorname{div} \widehat{f} \geq \sum_{\Delta_{\widehat{X}} \subseteq \operatorname{div} \widehat{f}} \frac{\operatorname{mult}_{\Delta_{\widehat{X}}} \gamma^* \Delta_X}{\operatorname{mult}_{\mathcal{C},\Delta_X} D} \cdot \Delta_{\widehat{X}},$$

where $\Delta_X := \big(\gamma_* \Delta_{\widehat{X}}\big)_{\mathrm{red}}$ and $(\text{finite})/\infty = 0$.

Remark 3.7. Notice that (3.6.2) says nothing about the vanishing order of $\widehat{f}$ along divisors $\Delta_{\widehat{X}}$ that lie over the support of $\lfloor D \rfloor$. In a similar vein, (3.6.3) says nothing about the vanishing order of $\widehat{f}$ along divisors $\Delta_{\widehat{X}}$ outside the ramification locus.

3.3. **Alternative description.** In his seminal paper [Miy08], Miyaoka describes the $\mathcal{C}$-cotangent bundle in terms of the classic residue sequence for logarithmic differentials,

$$0 \to \Omega^1_X(\log \lfloor D \rfloor) \to \Omega^1_X(\log D) \xrightarrow{\text{residue}} \bigoplus_{i \,|\, m_i < \infty} \mathcal{O}_{D_i} \to 0.$$

An elementary computation in local coordinates shows that Miyaoka's construction agrees with Definition 3.2 above.

[5] Recall from Notation 2.22 that the ramification divisor is reduced, so that 3.6.1 is a statement about the vanishing locus of $\widehat{f}$, but not about the order of vanishing.

Observation 3.8 (Miyaoka's description of the C-cotangent bundle, [Miy08, p. 412]). In Setting 3.1, the C-cotangent bundle $\Omega^1_{(X,D,\gamma)} \subseteq \gamma^*\Omega^1_X(\log D)$ equals the kernel of the following composed morphism,

$$\gamma^*\Omega^1_X(\log D) \xrightarrow{\gamma^*(\text{residue})} \bigoplus_{i \,|\, m_i<\infty} \gamma^*\mathscr{O}_{D_i} \overset{\text{Rem. 2.41}}{=} \bigoplus_{i \,|\, m_i<\infty} \mathscr{O}_{\gamma^* D_i} \xrightarrow{\text{Rem. 2.40}} \bigoplus_{i \,|\, m_i<\infty} \mathscr{O}_{\lceil \frac{1}{m_i}\cdot \gamma^* D_i \rceil}.$$

This describes the C-cotangent bundle by means of the following exact sequence,

$$0 \to \Omega^1_{(X,D,\gamma)} \to \gamma^*\Omega^1_X(\log D) \to \bigoplus_i \mathscr{O}_{\lceil \frac{1}{m_i}\cdot \gamma^* D_i \rceil} \to 0. \tag{3.8.1}$$

As before, the notation in (3.8.1) follows the convention that $\frac{1}{\infty} = 0$. □

3.4. **Inclusions.** The construction equips the bundles of adapted tensors with numerous inclusions that we use throughout the paper. The following observation summarizes the most important ones for later reference.

Observation 3.9 (Inclusions). Assume Setting 3.1. Given numbers $n, p \in \mathbb{N}^+$, there exist natural inclusions as follows,

$$\begin{tikzcd}
\operatorname{Sym}^n \Omega^p_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor) \arrow[r, hook] & \operatorname{Sym}^n \Omega^p_{\widehat{X}}(\log \gamma^* D) \\
& \gamma^* \operatorname{Sym}^n \Omega^p_X(\log D) \arrow[u, hook] \\
\operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)} \arrow[uu, hook, "\iota_{n,p}"] \arrow[r, equal] & \operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)} \arrow[u, hook] \\
\operatorname{Sym}^n \Omega^p_{(X,D,\gamma)} \arrow[u, hook] \arrow[r, equal] & \operatorname{Sym}^n \Omega^p_{(X,D,\gamma)} \arrow[u, hook] \\
\gamma^* \operatorname{Sym}^n \Omega^p_X \arrow[u, hook] \arrow[r, equal] & \gamma^* \operatorname{Sym}^n \Omega^p_X. \arrow[u, hook]
\end{tikzcd}$$

Note that all sheaves here are subsheaves of the quasi-coherent sheaf of meromorphic tensors on $\widehat{X}$, that is, $\mathscr{M}_{\widehat{X}} \otimes \operatorname{Sym}^n \Omega^p_{\widehat{X}}$. □

Observation 3.10 (Uniformization). Assume that the morphism $\gamma : \widehat{X} \to X$ of Setting 3.1 is an adapted cover and consider the morphisms

$$\iota_{\bullet,\bullet} : \operatorname{Sym}^\bullet_C \Omega^\bullet_{(X,D,\gamma)} \hookrightarrow \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}(\log \gamma^*\lfloor D\rfloor)$$

of Observation 3.9. Then, the following statements are equivalent.

(3.10.1) The morphism γ is a uniformization.
(3.10.2) There exist numbers $1 \le p \le \dim X$ and $1 \le n$ such that $\iota_{n,p}$ is isomorphic.
(3.10.3) For every pair of numbers $p, n \in \mathbb{N}^+$, the inclusion $\iota_{n,p}$ is isomorphic. □

Remark 3.11. Observation 3.10 does not hold without the assumption that γ is adapted. For a counterexample, observe that the identity map $\gamma := \operatorname{Id}_X$ almost never uniformizes. Yet, we have seen in Item (3.5.3) of Example 3.5 that $\Omega^p_{(X,D,\operatorname{Id}_X)} = \Omega^p_X(\log\lfloor D\rfloor)$ for every number p.

3.5. **Operations.** Among all meromorphic differential forms, logarithmic forms are characterized by the fact that the pole order does not change under exterior derivatives and wedge products. The exterior derivatives and wedge products on $\Omega^\bullet_{\widehat{X}}(\log \gamma^* D)$ therefore induce operations on the bundles of adapted differentials.

Observation 3.12 (Wedge products and exterior derivatives). In Setting 3.1, observe that the subsheaves

$$\Omega^{\bullet}_{(X,D,\gamma)} \overset{\iota_{1,\bullet}}{\subseteq} \Omega^{\bullet}_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big)$$

are closed under wedge products and exterior derivatives. Given numbers p and q, we obtain natural operations

$$\wedge : \Omega^p_{(X,D,\gamma)} \times \Omega^q_{(X,D,\gamma)} \to \Omega^{p+q}_{(X,D,\gamma)} \quad \text{and} \quad d : \Omega^p_{(X,D,\gamma)} \to \Omega^{p+1}_{(X,D,\gamma)}. \qquad \square$$

We turn to symmetric powers. The trivial observation that

$$\lfloor n_1 \cdot \gamma^*\{D\} \rfloor + \lfloor n_2 \cdot \gamma^*\{D\} \rfloor \le \lfloor (n_1+n_2) \cdot \gamma^*\{D\} \rfloor, \quad \text{for all } n_1, n_2 \in \mathbb{N}$$

allows defining symmetric products on the bundles of adapted tensors that are compatible with the products in the standard symmetric algebra $\operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}(\log \gamma^* D)$.

Observation 3.13 (Symmetric multiplication). In Setting 3.1, observe that the subsheaves

$$\operatorname{Sym}^\bullet_C \Omega^\bullet_{(X,D,\gamma)} \subseteq \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big)$$

are closed under symmetric multiplication. Given numbers p, n_1, $n_2 \in \mathbb{N}^+$, we obtain natural maps

$$\operatorname{Sym}^{n_1}_C \Omega^p_{(X,D,\gamma)} \times \operatorname{Sym}^{n_2}_C \Omega^p_{(X,D,\gamma)} \to \operatorname{Sym}^{n_1+n_2}_C \Omega^p_{(X,D,\gamma)}$$

and

$$\operatorname{Sym}^{n_1} \operatorname{Sym}^{n_2}_C \Omega^p_{(X,D,\gamma)} \hookrightarrow \operatorname{Sym}^{n_1 \cdot n_2}_C \Omega^p_{(X,D,\gamma)}. \qquad \square$$

Observation 3.14 (Adapted tensors on adapted covers). If the q-morphism γ of Setting 3.1 is adapted, then $\gamma^*\{D\}$ is integral and

$$\operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)} = \operatorname{Sym}^n \Omega^p_{(X,D,\gamma)}.$$

In particular, we find that the symmetric multiplication maps of Observation 3.13 are surjective. $\square$

3.6. **Functoriality.** Given a C-pair (X,D) and two covers $\widehat{Y}_1 \twoheadrightarrow X$ and $\widehat{Y}_2 \twoheadrightarrow X$, one would often like to compare adapted tensors on $\widehat{Y}_1$ with those on $\widehat{Y}_2$. Typically, this amounts to choosing one cover $\widehat{X} \twoheadrightarrow X$ that dominates both $\widehat{Y}_\bullet$, and then comparing the adapted tensors on $\widehat{Y}_\bullet$ with those on $\widehat{X}$. The following observation yields the necessary comparison morphisms.

Observation 3.15 (Functoriality in q-morphisms). In the Setting 3.1, assume that the morphism γ factors into a sequence of q-morphisms,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & \widehat{Y} \arrow[r, "\beta"'] & X, \end{tikzcd}$$

where $\big(\widehat{Y}, \beta^* D\big)$ is likewise nc. Given numbers n, $p \in \mathbb{N}^+$, there exists a commutative diagram as follows,

$$\begin{tikzcd} \alpha^* \operatorname{Sym}^n \Omega^p_{\widehat{Y}}(\log \beta^* \lfloor D \rfloor) \arrow[r, hook, "d\,\alpha"] & \operatorname{Sym}^n \Omega^p_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big) \\ \alpha^* \operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)} \arrow[u, hook, "\text{inclusion } \alpha^*(\iota_{n,p})"] \arrow[r, hook] & \operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)}. \arrow[u, hook, "\text{inclusion } \iota_{n,p}"'] \end{tikzcd}$$

$\square$

Observation 3.16 (Functoriality in adapted morphisms). If the morphism β of Observation 3.15 is adapted for (X,D), then γ is likewise adapted. The natural morphisms $\alpha^* \operatorname{Sym}^\bullet_C \Omega^\bullet_{(X,D,\beta)} \hookrightarrow \operatorname{Sym}^\bullet_C \Omega^\bullet_{(X,D,\gamma)}$ are isomorphic in this case. $\square$

Observation 3.15 asserts that the pull-back of an adapted tensor on $\widehat{Y}$ is an adapted tensor on $\widehat{X}$. For future applications, the following observation notes that the converse is also true: A meromorphic tensor on $\widehat{Y}$ is adapted *if and only if* its pull-back is adapted on $\widehat{X}$. The proof is an exercise in "pull-back and round up/round down" and uses the classic fact that a tensor has logarithmic poles if and only if its pull-back has logarithmic poles.

Observation 3.17 (Test for adapted tensors, compare Lemma 2.23). In the setting of Observation 3.15, let n and $p \in \mathbb{N}^+$ be two numbers, let $U \subseteq \operatorname{img} \alpha \subseteq \widehat{Y}$ be open and let $\sigma \in \left(\mathscr{M}_{\widehat{Y}} \otimes \operatorname{Sym}^n \Omega^p_{\widehat{Y}}\right)(U)$ be any meromorphic tensor on U. Then, the following are equivalent.

(3.17.1) The section σ is an adapted tensor. More precisely: the meromorphic tensor σ is a section of the subsheaf $\operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)} \subseteq \mathscr{M}_{\widehat{Y}} \otimes \operatorname{Sym}^n \Omega^p_{\widehat{Y}}$.

(3.17.2) The pull-back of σ is an adapted tensor. More precisely: the meromorphic tensor $(d\,\alpha)(\sigma)$ is a section of the subsheaf $\operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)} \subseteq \mathscr{M}_{\widehat{X}} \otimes \operatorname{Sym}^n \Omega^p_{\widehat{X}}$. □

Consequence 3.18 (Trace morphism). In the setting of Observation 3.15, the trace map

$$\alpha_* \Omega^\bullet_{\widehat{X}}(\log \gamma^* D) \xrightarrow{\text{trace}} \Omega^\bullet_{\widehat{Y}}(\log \beta^* D)$$

maps adapted differentials to adapted differentials. More precisely, there exist commutative diagrams as follows,

$$\begin{tikzcd}
\alpha_* \Omega^\bullet_{(X,D,\gamma)} \arrow[r, hook, "\text{Obs. 3.9}"] \arrow[d, "\text{restr. of trace}"'] & \alpha_* \Omega^\bullet_{\widehat{X}}(\log \gamma^* D) \arrow[d, "\text{trace}"] \\
\Omega^\bullet_{(X,D,\beta)} \arrow[r, hook, "\text{Obs. 3.9}"'] & \Omega^\bullet_{\widehat{Y}}(\log \beta^* D)
\end{tikzcd}$$

□

We remark that Consequence 3.18 has no analogue for higher-order tensors. Already in the simplest case where $\gamma : \mathbb{A}^1 \to \mathbb{A}^1$ is a uniformization of the pair $(X, D) = \left(\mathbb{A}^1, \frac{1}{2} \cdot \{0\}\right)$ that factorizes as

$$\underbrace{\mathbb{A}^1}_{=\widehat{X}} \xrightarrow[z \mapsto z^2]{\alpha = \gamma} \underbrace{\mathbb{A}^1}_{=\widehat{Y}} \xrightarrow{\beta = \mathrm{Id}} \underbrace{\mathbb{A}^1}_{=X},$$

the Galois-invariant two-tensor

$$dz \cdot dz \in H^0\left(\widehat{X}, \operatorname{Sym}^2_C \Omega^1_{(X,D,\gamma)}\right) = H^0\left(\mathbb{A}^1, \operatorname{Sym}^2 \Omega^1_{\mathbb{A}^1}\right)$$

does not induce any section of $\operatorname{Sym}^2_C \Omega^1_{(X,D,\beta)} = \operatorname{Sym}^2 \Omega^1_{\mathbb{A}^1}$.

3.7. **Galois linearization.** All sheaves that we have discussed in Definition 3.2 are linearized with respect to the action of the relative automorphism group $\operatorname{Aut}_{\mathscr{O}}(\widehat{X}/X)$. If the morphism γ is Galois, then all sheaves are Galois-linearized. We refer the reader to [GKKP11, Appendix A and references there] for more on G-sheaves and G-invariant push-forward.

Observation 3.19 (Linearisation). Assume Setting 3.1 and write $G := \operatorname{Aut}_{\mathscr{O}}(\widehat{X}/X)$ for the relative automorphism group. Then, all sheaves $\operatorname{Sym}^\bullet_C \Omega^\bullet_{(X,D,\gamma)}$ of Definition 3.2 carry natural G-linearisations that are compatible with the natural G-linearisations of

$$\operatorname{Sym}^n \gamma^* \Omega^p_X, \quad \gamma^* \operatorname{Sym}^n \Omega^p_X(\log D), \quad \operatorname{Sym}^n \Omega^p_{\widehat{X}}(\log \gamma^* D),$$
$$\operatorname{Sym}^n \Omega^p_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor).$$

The inclusions of Observation 3.9 are morphisms of G-sheaves. □

Observation 3.19 applies most prominently in settings where γ is Galois. It also applies in the setting of Observation 3.15, effectively allowing us to compare adapted tensors on $\widehat{X}$ with those on $\widehat{Y}$.

Lemma 3.20 (Invariant push-forward)**.** *In the setting of Observation 3.15, assume that the q-morphisms α, β and γ are covers, and that α is Galois with group G. Given numbers n, $p \in \mathbb{N}^+$, the natural morphism between G-invariant push-forward sheaves induced by the bottom row of the commutative diagram in Observation 3.15,*

$$\operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)} = \Bigl(\alpha_* \alpha^* \operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)}\Bigr)^G \hookrightarrow \Bigl(\alpha_* \operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)}\Bigr)^G,$$

is isomorphic.

Proof. Given an open set $U \subseteq \widehat{Y}$ and a G-invariant adapted tensor on $\widehat{X}$,

$$\sigma \in \operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)}(\alpha^{-1}U),$$

we need to find an adapted tensor $\tau \in \operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)}(U)$ whose pull-back equals σ. To this end, consider the inclusion

$$\operatorname{Sym}^n_C \Omega^p_{(X,D,\gamma)} \hookrightarrow \gamma^* \operatorname{Sym}^n \Omega^p_X(\log D) \tag{3.20.1}$$

discussed in Observation 3.9. We have remarked in Observation 3.19 that (3.20.1) is an inclusion of G-linearized sheaves; the G-invariant adapted tensor σ will therefore define a G-invariant section

$$\sigma' \in \gamma^* \operatorname{Sym}^n \Omega^p_X(\log D)(\alpha^{-1}U) = \alpha^*\beta^* \operatorname{Sym}^n \Omega^p_X(\log D)(\alpha^{-1}U).$$

In this G-invariant setting, it immediately equips us with an associated section

$$\tau' \in \beta^* \operatorname{Sym}^n \Omega^p_X(\log D)(U),$$

whose pull-back equals σ'. In order to conclude, it remains to show that τ' is an adapted tensor. Equivalently: It remains to show that τ' really a section of the subsheaf

$$\operatorname{Sym}^n_C \Omega^p_{(X,D,\beta)} \hookrightarrow \beta^* \operatorname{Sym}^n \Omega^p_X(\log D).$$

This is exactly the implication (3.17.2) ⇒ (3.17.1) of Observation 3.17 above. □

3.8. **Chern classes.** Observation 3.8 describes the C-cotangent bundle by means of an exact sequence that allows computing Chern classes. For simplicity, we restrict ourselves to the compact setting, where Chern classes for coherent analytic sheaves can be defined in rational cohomology, as explained [TT86] and briefly recalled in [Gri10, Sect. 1]. In settings where more general or more refined classes exist, the computations described here will work without change.

Observation 3.21 (Total Chern class of the C-cotangent bundle). In Setting 3.1, assume that the q-morphism γ is adapted. Then,

$$\left\lceil \tfrac{1}{m_i} \cdot \gamma^* D_i \right\rceil = \tfrac{1}{m_i} \cdot \gamma^* D_i, \quad \text{for every } i.$$

If $\widehat{X}$ and X are compact, then an elementary computation applying the Whitney formula to Sequence (3.8.1) reveals the total Chern class of $\Omega^1_{(X,D,\gamma)}$ as

$$\begin{aligned} c\Bigl(\Omega^1_{(X,D,\gamma)}\Bigr) &= c\Bigl(\gamma^*\Omega^1_X(\log D)\Bigr) \cdot \prod_{i \,|\, m_i < \infty} c\Bigl(\mathscr{O}_{\frac{1}{m_i}\cdot \gamma^* D_i}\Bigr)^{-1} \\ &= \gamma^* \left(c\bigl(\Omega^1_X\bigr) \cdot \prod_i \left(\tfrac{m_i - 1}{m_i} \cdot c\bigl(\mathscr{O}_{D_i}\bigr) + \tfrac{1}{m_i} \right) \right) \in H^*\bigl(\widehat{X}, \mathbb{Q}\bigr). \end{aligned}$$

In particular, we find that

$$c_1\big(\Omega^1_{(X,D,\gamma)}\big) = \gamma^* c_1\big(K_X + D\big) \in H^*\big(\widehat{X},\, \mathbb{Q}\big).$$

Definition 3.22 (Total $\mathcal{C}$-Chern class of $\mathcal{C}$-cotangent bundle)**.** *If (X, D) is a compact nc $\mathcal{C}$-pair, write*

$$c\Big(\Omega^1_{(X,D)}\Big) := c\big(\Omega^1_X\big) \cdot \prod_i \Big(\tfrac{m_i-1}{m_i} \cdot c\big(\mathscr{O}_{D_i}\big) + \tfrac{1}{m_i}\Big) \in H^*\big(X,\, \mathbb{Q}\big)$$

and refer to this quantity as the total $\mathcal{C}$-Chern class *of the $\mathcal{C}$-cotangent bundle for (X, D).*

Chern classes of $\mathcal{C}$-cotangent bundles have been studied at length in the literature. While the surface case has already been considered in Miyaoka's classic paper [Miy08, Sect. 3], generalizations to higher dimensions appear throughout the recent literature, including [GT22, Sects. 2 and 3] and [CDR20, Sect. 2.6].

3.9. **Residue sequences for the $\mathcal{C}$-cotangent bundle.** In view of Miyaoka's description of the $\mathcal{C}$-cotangent bundle, it is perhaps not surprising that the classic residue– and normal bundle sequences for logarithmic differentials have direct counterparts in the setting of $\mathcal{C}$-pairs.

Observation 3.23 ($\mathcal{C}$-residue sequence). In Setting 3.1, assume that the q-morphism γ is adapted. The alternative description of the $\mathcal{C}$-cotangent sheaf in Observation 3.8 expands to the following commutative diagram with exact rows and columns,

$$\begin{tikzcd}
\gamma^*\Omega^1_X(\log\lfloor D\rfloor) \arrow[r, hook] \arrow[d, equal] & \Omega^1_{(X,D,\gamma)} \arrow[r, two heads] \arrow[d, hook] & \bigoplus_{i \mid m_i<\infty} \mathscr{I}_{\frac{1}{m_i}\cdot\gamma^*D_i} \big/ \mathscr{I}_{\gamma^*D_i} \arrow[d, hook] \\
\gamma^*\Omega^1_X(\log\lfloor D\rfloor) \arrow[r, hook] \arrow[d, two heads] & \gamma^*\Omega^1_X(\log D) \arrow[r, two heads] \arrow[d, two heads] & \bigoplus_{i \mid m_i<\infty} \mathscr{O}_{\gamma^*D_i} \arrow[d, two heads] \\
0 \arrow[r, hook] & \bigoplus_{i \mid m_i<\infty} \mathscr{O}_{\lceil\frac{1}{m_i}\cdot\gamma^*D_i\rceil} \arrow[r, equal] & \bigoplus_{i \mid m_i<\infty} \mathscr{O}_{\frac{1}{m_i}\cdot\gamma^*D_i}.
\end{tikzcd}$$

Its top row is called the *$\mathcal{C}$-residue sequence* of the pair (X, D) and the q-morphism γ.

3.10. **Normal bundle sequences for the $\mathcal{C}$-cotangent bundle.** In contrast to the $\mathcal{C}$-residue sequence, the $\mathcal{C}$-normal bundle sequences are a little more delicate to write down, because we need to choose compatible submanifolds $Y \subsetneq X$ and $\widehat{Y} \subsetneq \widehat{X}$. For simplicity, we stick to the simplest setting where Y and $\widehat{Y}$ are of pure codimension one, and consider the cases where $Y \not\subset \operatorname{supp} D$ and $Y \subset \operatorname{supp} D$ separately. The reader will observe how the $\mathcal{C}$-normal bundle sequences in Observations 3.25 and 3.26 interpolate between the classic and the logarithmic case.

Setting 3.24 ($\mathcal{C}$-normal bundle sequence). In Setting 3.1, assume that the q-morphism γ is adapted. Let $Y \subsetneq X$ and $\widehat{Y} \subseteq \gamma^{-1}(Y) \subsetneq \widehat{X}$ be smooth prime divisors, such that $Y + D$ and $\widehat{Y} + \gamma^* D$ have nc support in X and $\widehat{X}$, respectively.

Observation 3.25 ($\mathcal{C}$-Normal bundle sequence, I). In Setting 3.24, assume that $Y \not\subset \operatorname{supp} D$. Write

$$D_Y := D|_Y \in \mathbb{Q}\operatorname{Div}(Y)$$

and observe that (Y, D_Y) is again a $\mathcal{C}$-pair. The restricted morphism $\gamma|_{\widehat{Y}} : \widehat{Y} \to Y$ is again an adapted q-morphism for the pair (Y, D_Y) that satisfies the assumptions of Setting 3.1, so that a well-defined $\mathcal{C}$-cotangent bundle $\Omega^1_{(Y,D_Y,\gamma|_{\widehat{Y}})}$ exists. Further, there exists a natural sequence

$$0 \to \mathscr{O}_{\widehat{X}}\big(\gamma^* Y\big)|_{\widehat{Y}} \to \Omega^1_{(X,D,\gamma)}|_{\widehat{Y}} \to \Omega^1_{(Y,D_Y,\gamma|_{\widehat{Y}})} \to 0. \tag{3.25.1}$$

Observation 3.26 (C-Normal bundle sequence, II). In Setting 3.24, assume that $Y \subseteq \operatorname{supp} D$, so that $Y = D_{i_0}$ for one index i_0. Write

$$D_Y := \left(D - \tfrac{m_{i_0}-1}{m_{i_0}} D_{i_0}\right)\Big|_Y \in \mathbb{Q}\operatorname{Div}(Y)$$

and observe that (Y, D_Y) is again a C-pair. The restricted morphism $\gamma|_{\widehat{Y}} : \widehat{Y} \to Y$ is again an adapted q-morphism for the pair (Y, D_Y) that satisfies the assumptions of Setting 3.1, so that a well-defined C-cotangent bundle $\Omega^1_{(Y,D_Y,\gamma|_{\widehat{Y}})}$ exists. With the understanding that $\frac{1}{\infty} = 0$, there exists a natural sequence

$$0 \to \mathscr{O}_{\widehat{X}}\big(\tfrac{1}{m_{i_0}} \cdot \gamma^* D_{i_0}\big)|_{\widehat{Y}} \to \Omega^1_{(X,D,\gamma)}|_{\widehat{Y}} \to \Omega^1_{(Y,D_Y,\gamma|_{\widehat{Y}})} \to 0. \tag{3.26.1}$$

Notation 3.27 (C-normal bundle sequence). We refer to Sequences (3.25.1) and (3.26.1) as *C-normal bundle sequences* of the pair (X, D).

4. Adapted reflexive tensors

Given a C-pair (X, D) and a q-morphism $\gamma : \widehat{X} \to X$, Section 3 defined adapted tensors on $\widehat{X}$, assuming that the spaces X and $\widehat{X}$ are smooth and that the divisors D and $\gamma^* D$ have normal crossing support. While we hope that the reader finds the resulting notions interesting, we have to admit that the strong smoothness assumptions limit the theory's usefulness in practise.

- Pairs (X, D) that appear in classification and birational geometry are hardly ever nc. Practically relevant pairs will typically be klt and might be locally uniformizable at best.
- Even if (X, D) is nc, most of the covering spaces that one might naturally consider will typically be singular. Observe that smoothness is not preserved by fibre-product constructions.

This section extends the constructions of Section 3 to the singular case, replacing "adapted tensors" by the "adapted reflexive tensors" that we define in the next step. The construction also generalize the "sheaves of reflexive differentials" of Notation 2.6 that have been useful in the study of singular varieties that appear in Minimal Model Theory, [GKKP11, KS21].

4.1. **Definition and first examples.**

The present Section 4 works in the following setting and uses the following notation.

Setting 4.1. Let (X, D) be a C-pair as in Definition 2.24, where D is written as $\sum_i \frac{m_i-1}{m_i} \cdot D_i$. Let $\gamma : \widehat{X} \to X$ be a q-morphism.

Notation 4.2. In Setting 4.1, recall from Reminder 2.19 that $\operatorname{img}\gamma \subseteq X$ is open. Let $X^+ \subseteq \operatorname{img}\gamma$ be the maximal open set such that the pairs (X, D) and $(\widehat{X}, \gamma^* D)$ are nc over X^+. Set

$$D^+ := D \cap X^+ \in \mathbb{Q}\operatorname{Div}(X^+) \quad \text{and} \quad \widehat{X}^+ := \gamma^{-1}(X^+).$$

Observe that the subset $\widehat{X}^+ \subseteq \widehat{X}$ is big and consider the restriction $\gamma^+ : \widehat{X}^+ \to X^+$ and the inclusion $\iota : \widehat{X}^+ \to \widehat{X}$.

Observe that the pair (X^+, D^+) and the morphism γ^+ satisfy the assumptions made in Setting 3.1 above. Definition 3.2 therefore equips us with bundles $\operatorname{Sym}^\bullet_C \Omega^\bullet_{(X^+,D^+,\gamma^+)}$ defined on $\widehat{X}^+$. We extend these bundles from $\widehat{X}^+$ to a quasi-coherent sheaves that are defined on all of $\widehat{X}$.

Definition 4.3 (Adapted reflexive tensors differentials, compare Definition 3.2)**.** *Assume Setting 4.1. Given numbers $n, p \in \mathbb{N}^+$, define the* sheaf of adapted reflexive (n, p)-tensors *as*

$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} := \iota_* \operatorname{Sym}^n_C \Omega^p_{(X^+,D^+,\gamma^+)}.$$

Collectively, we refer to $\operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D,\gamma)}$ *as the sheaves of adapted reflexive tensors.*

Definition 4.4 (Adapted reflexive differentials, compare Definition 3.3)**.** *Assume Setting 4.1. Given a number $p \in \mathbb{N}^+$, define the* sheaf of adapted reflexive p-forms *as*

$$\Omega^{[p]}_{(X,D,\gamma)} := \operatorname{Sym}^{[1]}_C \Omega^{[p]}_{(X,D,\gamma)}.$$

Collectively, we refer to $\Omega^{[\bullet]}_{(X,D,\gamma)}$ *as the* sheaves of adapted reflexive differentials. *The sheaf* $\Omega^{[1]}_{(X,D,\gamma)}$ *is called* C-cotangent sheaf.

If (X, D) and $(\widehat{X}, \gamma^* D)$ are nc, then the sheaves of adapted reflexive tensors agree with the bundles constructed in Definition 3.2 and are thus locally free. In general, we show that the sheaves of adapted reflexive tensors are reflexive.

Proposition 4.5 (Reflexivity, compare Observation 3.4)**.** *The sheaves* $\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)}$ *of Definition 4.3 are reflexive.*

The reader coming from algebraic geometry might find the following proof of Proposition 4.5 surprisingly complicated. We recall that in the analytic setting, vector bundles on big open subsets can generally *not* be extended to coherent sheaves on the whole space and refer the reader to [Ser66, p. 372] for an elementary yet sobering example.

Proof of Proposition 4.5. It follows from Observation 2.15 that the open subset $\widehat{X}^+ \subseteq \widehat{X}$ of Notation 4.2 is big. With that in place, recall [Ser66, Prop. 7]: To prove that the sheaves $\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)}$ are reflexive, it suffices to find coherent sheaves $\mathscr{F}_{n,p}$ on $\widehat{X}$ whose restrictions to $\widehat{X}^+$ agree with the bundles of (n, p)-tensors,

$$\mathscr{F}_{n,p}|_{\widehat{X}^+} \cong \operatorname{Sym}^n_C \Omega^p_{(X^+,D^+,\gamma^+)}. \tag{4.5.1}$$

In order to construct $\mathscr{F}_{n,p}$, choose a strong log resolution[6] $\pi : Y \to X$ of the pair (X, D) and consider the strict transform $\Delta := \pi^{-1}_* D$. Choosing a strong log resolution of the normalized fibre product, we obtain a commutative diagram of dominant morphisms between normal analytic varieties,

$$\begin{tikzcd} \widehat{Y} \arrow[rr, "\widehat{\pi}\text{, proper birational}"] \arrow[d, "\delta"'] & & \widehat{X} \arrow[d, "\gamma"] \\ Y \arrow[rr, "\pi\text{, strong log resolution}"'] & & X, \end{tikzcd}$$

where π and $\widehat{\pi}$ are isomorphic over X^+ and $\widehat{X}^+$, respectively. In analogy to Definition 3.2, write

$$\mathscr{A}'_{n,p} := \mathscr{O}_{\widehat{Y}}\big(\lfloor n \cdot \delta^* \{\Delta\} \rfloor\big) \otimes \delta^* \operatorname{Sym}^n \Omega^p_Y(\log \lfloor \Delta \rfloor)$$
$$\mathscr{B}'_{n,p} := \operatorname{Sym}^n \Omega^p_{\widehat{Y}}(\log \gamma^* \Delta).$$

Observe that both are subsheaves of $\mathscr{O}_{\widehat{Y}}(\delta^* \Delta) \otimes (\delta^* \operatorname{Sym}^n \Omega^p_Y)$ and define

$$\mathscr{F}'_{n,p} := \mathscr{A}'_{n,p} \cap \mathscr{B}'_{n,p}.$$

[6] A strong log resolution is a proper morphism $\pi : Y \to X$ where Y is smooth, where π is isomorphic over the maximal open set where (X, D) is nc, and where the π-exceptional set $E \subset Y$ and $E \cup \pi^{-1}(X \setminus U)$ are both of pure codimension one with nc support.

We can then take $\mathscr{F}_{n,p} := \widehat{\pi}_* \mathscr{F}'_{n,p}$, which is coherent because $\widehat{\pi}$ is proper. Since π and $\widehat{\pi}$ are isomorphic over X^+ and $\widehat{X}^+$, condition (4.5.1) holds by construction. □

In analogy with Example 3.5, we highlight a few special cases where the sheaves of adapted reflexive tensors take a particularly simple form.

Example 4.6 (Special cases, compare Example 3.5). In Setting 4.1, assume that numbers $n, p \in \mathbb{N}^+$ are given.

(4.6.1) If $p = \dim X$, then

$$\operatorname{Sym}^{[n]}_C \Omega^{[\dim X]}_{(X,D,\gamma)} = \Big(\big(\gamma^{[*]} \omega_X^{\otimes n}\big) \otimes \mathscr{O}_{\widehat{X}}\big(\lfloor n \cdot \gamma^* D \rfloor\big)\Big)^{\vee\vee}.$$

(4.6.2) If $D = 0$, then $\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,0,\gamma)} = \gamma^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_X$.

(4.6.3) If $\gamma = \mathrm{Id}_X$, then $\Omega^{[p]}_{(X,D,\mathrm{Id}_X)} = \Omega^{[p]}_X(\log \lfloor D \rfloor)$.

(4.6.4) If γ is strongly adapted and $\operatorname{Branch}(\gamma) \subseteq \operatorname{supp} D$, then

$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} = \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big).$$

(4.6.5) If γ uniformizes, then

$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} = \operatorname{Sym}^{n} \Omega^{p}_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big).$$

Remark 4.7 (Reflexive hull in (4.6.1)). The double dual on the right side of (4.6.1) is necessary, as the tensor product of two reflexive sheaves will generally not be reflexive and might even contain torsion. If a canonical divisor K_X exists on X, then (4.6.1) simplifies to

$$\operatorname{Sym}^{[n]}_C \Omega^{[\dim X]}_{(X,D,\gamma)} = \mathscr{O}_{\widehat{X}}\big(\lfloor n \cdot \gamma^*(K_X + D) \rfloor\big).$$

4.2. **Inclusions.** By construction, the observations in Section 3 have direct analogues for the sheaves of adapted reflexive tensors. For later reference and for the reader's convenience, we include full statements, even though the text does become somewhat repetitive and perhaps a little tiring.

Observation 4.8 (Inclusions, compare Observation 3.9). Assume Setting 4.1. Given numbers $n, p \in \mathbb{N}^+$, there exist natural inclusions as follows,

$$\begin{tikzcd}
\operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big(\log \gamma^* \lfloor D \rfloor\big) \arrow[r, hook] & \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big(\log \gamma^* D\big) \\
 & \gamma^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_X(\log D) \arrow[u, hook] \\
\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} \arrow[uu, hook, "\iota_{n,p}"] \arrow[r, equal] & \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} \arrow[u, hook] \\
\operatorname{Sym}^{[n]} \Omega^{[p]}_{(X,D,\gamma)} \arrow[u, hook] \arrow[r, equal] & \operatorname{Sym}^{[n]} \Omega^{[p]}_{(X,D,\gamma)} \arrow[u, hook] \\
\gamma^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_X \arrow[u, hook] \arrow[r, equal] & \gamma^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_X .\arrow[u, hook]
\end{tikzcd}$$

Note that all sheaves here are subsheaves of the quasi-coherent sheaf of meromorphic reflexive tensors on $\widehat{X}$, that is, $\mathscr{M}_{\widehat{X}} \otimes \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}$.

Proof. Like nearly every other statement in the remainder of the present Section 4, the proof follows from the observation that the subset $\widehat{X}^+ \subseteq \widehat{X}$ introduced in Setting 4.1 and Notation 4.2 is big and that $\widehat{X}$ is normal. If $\mathscr{A}$ and $\mathscr{B}$ are reflexive sheaves on $\widehat{X}$, this implies that the natural restriction morphisms

$$H^0\big(\widehat{X},\,\mathscr{A}\big) \to H^0\big(\widehat{X}^+,\,\mathscr{A}|_{\widehat{X}^+}\big) \quad\text{and}\quad \operatorname{Hom}_{\widehat{X}}\big(\mathscr{A},\,\mathscr{B}\big) \to \operatorname{Hom}_{\widehat{X}^+}\big(\mathscr{A}|_{\widehat{X}^+},\,\mathscr{B}|_{\widehat{X}^+}\big)$$

are isomorphic, [BS76, Cor. 3.15] and [Ser66]. Observation 3.9 therefore implies the claim. □

Observation 4.9 (Uniformization, compare Observation 3.10). Assume that the morphism $\gamma : \widehat{X} \to X$ of Setting 4.1 is an adapted cover and that the pair $\big(\widehat{X}, (\gamma^*\lfloor D\rfloor)_{\mathrm{reg}}\big)$ is nc. Consider the morphisms

$$\iota_{\bullet,\bullet} : \operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D,\gamma)} \hookrightarrow \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_{\widehat{X}}\big(\log \gamma^*\lfloor D\rfloor\big) \tag{4.9.1}$$

of Observation 4.8. Then, the following statements are equivalent.

(4.9.2) The morphism γ is a uniformization.
(4.9.3) There exist numbers $1 \le p \le \dim X$ and $1 \le n$ such that $\iota_{n,p}$ is isomorphic.
(4.9.4) For every pair of numbers $p, n \in \mathbb{N}^+$, the inclusion $\iota_{n,p}$ is isomorphic.

Proof. By construction of the morphism (4.9.1), Items (4.9.3) and Items (4.9.4) is equivalent the analogous statements for the morphisms

$$\iota^+_{\bullet,\bullet} : \operatorname{Sym}^{\bullet}_C \Omega^{\bullet}_{(X^+,D^+,\gamma^+)} \hookrightarrow \operatorname{Sym}^{\bullet} \Omega^{\bullet}_{\widehat{X}^+}\big(\log(\gamma^+)^*\lfloor D^+\rfloor\big)$$

discussed in Observation 3.9. Observation 3.10 therefore implies that Items (4.9.3) and (4.9.4) are each equivalent to the assertion that γ^+ uniformizes. On the other hand, the assumption that $\big(\widehat{X}, (\gamma^*\lfloor D\rfloor)_{\mathrm{reg}}\big)$ is nc allows reformulating (4.9.2) as follows,

$$\begin{aligned} \gamma \text{ uniformizes } (X,D) &\Leftrightarrow \operatorname{Branch}\gamma \subseteq \operatorname{supp} D \text{ and } \gamma \text{ strongly adapted} && \text{Def. 2.28} \\ &\Leftrightarrow \operatorname{Branch}\gamma^+ \subseteq \operatorname{supp} D^+ \text{ and } \gamma^+ \text{strongly adapted} && \widehat{X}^+ \subseteq \widehat{X} \text{ big} \\ &\Leftrightarrow \gamma^+ \text{ uniformizes } (X^+, D^+) && \text{Def. 2.28.} \end{aligned}$$

For the second equivalence, recall from Notation 2.22 that $\operatorname{Branch}\gamma$ refers to the branch *divisor* and not to the branch *locus*, which might contain components of high codimension. □

4.3. **Operations.** With the minor difference highlighted in Warning 4.13 below, the operations on adapted tensors introduced in Section 3.5 extend to identical operations on adapted reflexive tensors.

Observation 4.10 (Reflexive wedge products and exterior derivatives, compare Observation 3.12). In Setting 4.1, observe that the subsheaves

$$\Omega^{[\bullet]}_{(X,D,\gamma)} \overset{\iota_{1,\bullet}}{\subseteq} \Omega^{[\bullet]}_{\widehat{X}}\big(\log \gamma^*\lfloor D\rfloor\big)$$

are closed under reflexive wedge products and exterior derivations. Given numbers p and q, we obtain natural operations

$$\wedge : \Omega^{[p]}_{(X,D,\gamma)} \times \Omega^{[q]}_{(X,D,\gamma)} \to \Omega^{[p+q]}_{(X,D,\gamma)} \quad\text{and}\quad d : \Omega^{[p]}_{(X,D,\gamma)} \to \Omega^{[p+1]}_{(X,D,\gamma)}. \qquad \square$$

Observation 4.11 (Reflexive symmetric multiplication, compare Observation 3.13). In Setting 4.1, observe that the subsheaves

$$\operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D,\gamma)} \subseteq \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_{\widehat{X}}\big(\log \gamma^*\lfloor D\rfloor\big)$$

are closed under symmetric multiplication. Given numbers p, n_1, $n_2 \in \mathbb{N}^+$, we obtain natural maps

$$\operatorname{Sym}^{[n_1]}_C \Omega^{[p]}_{(X,D,\gamma)} \times \operatorname{Sym}^{[n_2]}_C \Omega^{[p]}_{(X,D,\gamma)} \to \operatorname{Sym}^{[n_1+n_2]}_C \Omega^{[p]}_{(X,D,\gamma)}$$

and

$$\operatorname{Sym}^{[n_1]} \operatorname{Sym}_C^{[n_2]} \Omega^{[p]}_{(X,D,\gamma)} \hookrightarrow \operatorname{Sym}_C^{[n_1\cdot n_2]} \Omega^{[p]}_{(X,D,\gamma)}. \qquad \square$$

Observation 4.12 (Adapted reflexive tensors on adapted covers, compare Observation 3.14). If the q-morphism γ of Setting 4.1 is adapted, then $\gamma^*\{D\}$ is integral and

$$\operatorname{Sym}_C^{[n]} \Omega^{p}_{(X,D,\gamma)} = \operatorname{Sym}^{[n]} \Omega^{[p]}_{(X,D,\gamma)} \quad \text{for all } p, n \in \mathbb{N}^+. \qquad \square$$

Warning 4.13 (No surjectivity in Observation 4.12). The sheaves $\Omega^{[\bullet]}_{(X,D,\gamma)}$ in Observation 4.12 need not be locally free. The natural morphisms

$$\operatorname{Sym}^\bullet \Omega^{[\bullet]}_{(X,D,\gamma)} \to \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\gamma)}$$

are neither injective nor surjective in general; notice that the left side might well contain torsion! In contrast to Observation 3.13, we cannot conclude that the symmetric multiplication maps of Observation 4.10 are surjective.

4.4. **Functoriality.** The functoriality statements of Section 3.6 also have direct analogues. In line with Warning 4.13 above, there is a caveat here, stemming from the fact that the reflexive hull construction does not commute with pull-back. We highlight this issue in Warning 4.16, as it will become central when we define and discuss morphisms of C-pairs in Section 7ff, in the second part of this paper.

Observation 4.14 (Functoriality in q-morphisms, compare Observation 3.15). In Setting 4.1, assume that the morphism γ factors into a sequence of q-morphisms,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & \widehat{Y} \arrow[r, "\beta"'] & X. \end{tikzcd}$$

Given numbers $n, p \in \mathbb{N}^+$, there exists a commutative diagram as follows,

$$\begin{tikzcd} \alpha^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{Y}}(\log \beta^* \lfloor D \rfloor) \arrow[r, hook, "\text{equals } d\,\alpha \text{ over nc locus}"] & \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor) \\ \alpha^{[*]} \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D,\beta)} \arrow[u, hook, "\text{inclusion } \alpha^{[*]}(\iota_{n,p})"] \arrow[r, hook] & \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D,\gamma)}. \arrow[u, hook, "\text{inclusion } \iota_{n,p}"'] \end{tikzcd}$$

$\square$

Observation 4.15 (Functoriality in adapted morphisms, compare Observation 3.16). If the morphism β of Observation 4.14 is adapted for (X, D), then γ is likewise adapted. The natural morphisms $\alpha^{[*]} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)} \hookrightarrow \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\gamma)}$ are isomorphic in this case. $\square$

Warning 4.16 (Reflexive pull-back in the functoriality statement). In the setting of Observation 4.14, there exist natural sheaf morphisms

$$\alpha^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)} \to \alpha^{[*]} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)} \tag{4.16.1}$$

that are however neither injective nor surjective in general; notice that $\alpha^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)}$ might well contain torsion! This will become important. For later reference, we note a few settings where (4.16.1) is isomorphic indeed.

(4.16.2) The morphism (4.16.1) is isomorphic if $\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)}$ is locally free.

(4.16.3) The morphism (4.16.1) is isomorphic if α is flat.

On the positive side, observe that if the q-morphism β is adapted and $\Omega^{[\bullet]}_{(X,D,\beta)}$ is locally free, then

$$\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)} \overset{\text{Obs. 4.12}}{=} \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_{(X,D,\beta)} \overset{\text{loc. free}}{=} \operatorname{Sym}^\bullet \Omega^{[\bullet]}_{(X,D,\beta)}$$

is locally free, so that Item (4.16.2) applies.

As in Section 3.6, we note that a partial converse of Observation 4.14 holds true.

Observation 4.17 (Test for adapted reflexive tensors, compare Observation 3.17). In the setting of Observation 4.14, let n and $p \in \mathbb{N}^+$ be two numbers, let $U \subseteq \operatorname{img}\alpha \subseteq \widehat{Y}$ be open and let $\sigma \in \left(\mathscr{M}_{\widehat{Y}} \otimes \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{Y}}\right)(U)$ be any meromorphic reflexive tensor on U. Then, the following are equivalent.

(4.17.1) The section σ is an adapted reflexive tensor. More precisely: the meromorphic reflexive tensor σ is a section of the subsheaf
$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)} \subseteq \mathscr{M}_{\widehat{Y}} \otimes \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{Y}}.$$

(4.17.2) The pull-back of σ is an adapted reflexive tensor. More precisely: the meromorphic reflexive tensor $(d\,\alpha)(\sigma)$ is a section of the subsheaf
$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} \subseteq \mathscr{M}_{\widehat{Y}} \otimes \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}. \qquad \square$$

Consequence 4.18 (Trace morphism, compare Consequence 3.18). In the setting of Observation 4.14, the trace map
$$\alpha_* \Omega^{[\bullet]}_{\widehat{X}}(\log \gamma^* D) \xrightarrow{\text{trace}} \Omega^{[\bullet]}_{\widehat{Y}}(\log \beta^* D)$$
maps adapted reflexive differentials to adapted reflexive differentials. More precisely, there exist commutative diagrams as follows,
$$\begin{tikzcd}
\alpha_* \Omega^{[\bullet]}_{(X,D,\gamma)} \arrow[r, hook, "\text{Obs. 4.8}"] \arrow[d, "\text{restr. of trace}"'] & \alpha_* \Omega^{[\bullet]}_{\widehat{X}}(\log \gamma^* D) \arrow[d, "\text{trace}"] \\
\Omega^{[\bullet]}_{(X,D,\beta)} \arrow[r, hook, "\text{Obs. 4.8}"'] & \Omega^{[\bullet]}_{\widehat{Y}}(\log \beta^* D)
\end{tikzcd}$$
$\square$

4.5. **Galois linearization.** Unsurprisingly, the linearization morphisms discussed in Section 3.7 also extend from $\widehat{X}^+$ to $\widehat{X}$.

Observation 4.19 (Linearisation, compare Observation 3.19). Assume Setting 4.1 and write $G := \operatorname{Aut}_{\mathscr{O}}(\widehat{X}/X)$ for the relative automorphism group. Then, all sheaves $\operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D,\gamma)}$ of Definition 4.3 carry natural G-linearisations that are compatible with the natural G-linearisations of
$$\operatorname{Sym}^{[n]} \gamma^{[*]} \Omega^{[p]}_X, \quad \gamma^{[*]} \operatorname{Sym}^{[n]} \Omega^{[p]}_X(\log D), \quad \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}(\log \gamma^* D),$$
$$\operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor).$$
The inclusions of Observation 4.8 are morphisms of G-sheaves. $\square$

Lemma 4.20 (Invariant push-forward, compare Lemma 3.20)**.** *In the setting of Observation 4.14, assume that the q-morphisms α, β and γ are covers, and that α is Galois with group G. Given numbers n, $p \in \mathbb{N}^+$, the natural morphism between G-invariant push-forward sheaves induced by the bottom row of the commutative diagram in Observation 4.14,*
$$\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)} = \left(\alpha_* \alpha^{[*]} \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\beta)}\right)^G \hookrightarrow \left(\alpha_* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)}\right)^G, \tag{4.20.1}$$
is isomorphic.

Proof. Lemma 3.20 guarantees that the sheaf morphism (4.20.1) is isomorphic over the big open set $\widehat{X}^+ \subseteq \widehat{X}$. On the other hand, recall[7] from [GKKP11, Lem. A.4] that the invariant push-forward sheaves in (4.20.1) are both reflexive. The isomorphism will therefore extend from $\widehat{X}^+$ to $\widehat{X}$. $\square$

[7]The paper [GKKP11] formulates this result in the algebraic setting. The proof of [GKKP11, Lem. A.4] works without change also for normal analytic varieties.

5. Pull-back over uniformizable pairs

5.1. **Motivation.** This section establishes pull-back properties of adapted reflexive tensors; these will be instrumental when we define "morphisms of C-pairs" in Sections 7–8 below. To motivate the somewhat technical discussion, let us first recall Reid's construction of the Albanese for projective varieties with rational singularities, [Rei83, Prop. 2.3] and [BS95, Sect. 2.4].

Reminder 5.1 (Albanese for algebraic varieties with rational singularities). Given a complex, projective variety $\widehat{X}$ with rational singularities, Reid considers a resolution of singularities, $\varphi : Y \to \widehat{X}$ and takes the Albanese of Y. The assumption that $\widehat{X}$ has rational singularities implies that all 1-differentials on Y are trivial on φ-fibres. This in turn yields a factorization,

$$
\begin{tikzcd}
Y \arrow[r, "\mathrm{alb}(Y)"] \arrow[d, "{\varphi,\ \text{resolution}}"'] & \mathrm{Alb}(Y) \\
\widehat{X}, \arrow[ur, bend right, "\exists!"'] &
\end{tikzcd}
\tag{5.1.1}
$$

and shows that $\mathrm{Alb}(Y)$ does not depend on the choice of the resolution. It is therefore reasonable to take $\mathrm{Alb}(Y)$ as the Albanese of $\widehat{X}$.

For the forthcoming construction of an "Albanese for C-pairs", we would like to emulate Reid's argument in a setting where $\widehat{X}$ is a cover of a locally uniformizable C-pair (X, D). But covers need not have rational singularities, so that we cannot expect a factorization of the Albanese as in (5.1.1) above! We will however show that differentials on Y that are *adapted* outside the φ-exceptional locus are trivial on φ-fibres. More generally, we show that any adapted differential on

$$\widehat{X}_{\mathrm{reg}} \cong Y \setminus \varphi\text{-exceptional set}$$

extends to a differential form on Y that is trivial on φ-fibres. To give an adapted differential on $\widehat{X}_{\mathrm{reg}}$ it is of course equivalent to give an adapted reflexive differential $\widehat{X}$. By the end of the day, we will thus construct a "pull-back map"

$$H^0\big(\widehat{X},\ \Omega^{[1]}_{(X,D,\bullet)}\big) \to H^0\big(Y,\ \Omega^1_Y\big).$$

This section aims to construct pull-back maps more generally, for arbitrary tensors and arbitrary morphisms φ from manifolds to $\widehat{X}$.

5.2. **Main results.** To formulate our results precisely and to set the stage for the remainder of the present section, consider the following situation.

Setting 5.2 (Smooth space mapping to cover of X). Let (X, D_X) be a locally uniformizable C-pair. Let (Y, D_Y) be a log pair. Assume that (Y, D_Y) is nc and consider a sequence of morphisms

$$Y \xrightarrow{\ \varphi\ } \widehat{X} \xrightarrow{\ \gamma,\ q\text{-morphism}\ } X,$$

where $\operatorname{supp} \varphi^*\gamma^*\lfloor D_X \rfloor \subseteq \operatorname{supp} D_Y$.

Remark 5.3. We do not assume that the variety $\widehat{X}$ of Setting 5.2 is smooth. The morphism φ may take its image in the singular locus of $\widehat{X}$.

Maintain Setting 5.2. Following ideas and methods of [Keb13], we aim to construct "natural" pull-back morphisms

$$d_C\varphi : \varphi^* \operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D_X,\gamma)} \to \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y) \tag{5.4.1}$$

that compare adapted reflexive tensors on $\widehat{X}$ with logarithmic Kähler tensors on the manifold Y.

Remark 5.5 (Pull-back for a uniformized variety). Assume Setting 5.2. If the q-morphism γ is a uniformization, then Item (4.6.5) of Example 4.6 identifies adapted reflexive tensors on $\widehat{X}$ with logarithmic Kähler tensors,

$$\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} = \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}(\log \gamma^* \lfloor D_X \rfloor).$$

The most natural choice for the pull-back morphisms (5.4.1) is the standard pull-back of logarithmic Kähler differentials and tensors.

Remark 5.6 (Optimality and possible generalizations). This section works in Setting 5.2, where (X, D) is locally uniformizable. If one is interested only in adapted reflexive differentials rather than adapted reflexive tensors, it is conceivable that a pull-back morphism as in (5.4.1) exists under less restrictive conditions. We discuss possible generalizations in Section 15.1 near the end of this paper.

5.3. **Construction of pull-back maps in the uniformizable case.** We begin with an explicit construction for a pull-back morphism, at least in the setting where X is uniformizable.

Construction 5.7 (Pull-back of sections in the uniformizable case). In Setting 5.2, assume that (X, D_X) is uniformizable. Choose a uniformization $u : X_u \twoheadrightarrow X$ and consider a diagram

$$\begin{tikzcd} \check{Y}^+ \arrow[r, "\check{\varphi}"] \arrow[d, two heads, "s^+"'] & \check{X} \arrow[r, "\check{\gamma}"] \arrow[d, two heads, "t"] & X_u \arrow[d, two heads, "u\text{, uniformization}"] \\ Y^+ \arrow[r, "\varphi^+ := \varphi|_{Y^+}"'] & \widehat{X} \arrow[r, "\gamma\text{, } q\text{-morphism}"'] & X, \end{tikzcd}$$

constructed as follows.

- Choose a component $\check{X}$ of the normalized fibre product $X_u \times_X \widehat{X}$.
- Choose a component $\check{Y}$ of the normalized fibre product $Y \times_{\widehat{X}} \check{X}$ and denote the natural morphism by $s : \check{Y} \to Y$
- Let $Y^+ \subseteq Y$ be the maximal open set over which $(\check{Y}, s^* D_Y)$ is nc and denote the preimage by $\check{Y}^+ := s^{-1}(Y^+)$.

To begin the construction in earnest, observe that there are natural morphisms,

$$\begin{aligned} (s^+)^*(\varphi^+)^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} &= \check{\varphi}^* t^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} && \text{commutativity} \\ &\to \check{\varphi}^* t^{[*]} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} && \text{natural} \\ &\to \check{\varphi}^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma\circ t)} && \text{Observation 4.14.} \end{aligned} \tag{5.7.1}$$

Secondly, recall

$$\begin{aligned} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma\circ t)} &= \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,u\circ\check{\gamma})} && \text{commutativity} \\ &= \check{\gamma}^{[*]} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,u)} && \text{Observation 4.15} \\ &= \check{\gamma}^* \operatorname{Sym}^\bullet \Omega^\bullet_{X_u}(\log u^* \lfloor D_X \rfloor) && \text{Example 4.6.} \end{aligned} \tag{5.7.2}$$

As a consequence, pull-back of logarithmic Kähler tensors yields natural embeddings

$$\begin{aligned} \check{\varphi}^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma\circ t)} &= (\check{\gamma} \circ \check{\varphi})^* \operatorname{Sym}^\bullet \Omega^\bullet_{X_u}(\log u^* \lfloor D_X \rfloor) && (5.7.2) \\ &\hookrightarrow \operatorname{Sym}^\bullet \Omega^\bullet_{\check{Y}^+}(\log s^* D_Y) && \text{pull-back} \\ &= (s^+)^* \operatorname{Sym}^\bullet \Omega^\bullet_{Y^+}(\log D_Y) && \text{branching of } s^+. \end{aligned} \tag{5.7.3}$$

Combining (5.7.1) with (5.7.3) and taking push-forward, we obtain maps as follows,

$$\begin{aligned}(\varphi^+)^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} &\to (s^+)_*(s^+)^*(\varphi^+)^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} && \text{natural}\\ &\to (s^+)_*(s^+)^* \operatorname{Sym}^\bullet \Omega^\bullet_{Y^+}(\log D_Y) && (5.7.3)\circ(5.7.1)\\ &\to \operatorname{Sym}^\bullet \Omega^\bullet_{Y^+}(\log D_Y) && \text{trace.}\end{aligned}$$

Given that $\Omega^\bullet_Y(\log D_Y)$ is locally free and $Y^+ \subseteq Y$ is big, these maps extend to the desired pull-back morphisms $d_C\varphi$ of the form promised in (5.4.1) above.

We leave the proof of the following fact to the reader.

Fact 5.8 (Canonicity). *The pull-back morphisms $d_C\varphi$ of Construction 5.7 do not depend on any of the choices made in the construction.* □

5.4. **Construction of pull-back maps in general.** Construction 5.7 evidently commutes with restrictions to open subsets of domain and target, which allows extending the setup from the uniformizable to the locally uniformizable case.

Fact 5.9 (Pull-back over locally uniformizable pairs). *In Setting 5.2, there exist unique sheaf morphisms*

$$d_C\varphi : \varphi^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)} \to \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y)$$

such that for every uniformizable open subset $X^+ \subseteq X$ with preimages $\widehat{X}^+ \subseteq \widehat{X}$ and $Y^+ \subseteq Y$, the restrictions $d_C\varphi|_{Y^+}$ agree with the pull-back morphisms $d_C\big(\varphi|_{Y^+}\big)$ of Construction 5.7. □

Definition 5.10 (Pull-back over locally uniformizable pairs). *We refer to the pull-back morphisms $d_C\bullet$ of Fact 5.9 as the* pull-back for adapted reflexive tensors over the locally uniformizable pair (X, D_X).

5.5. **Universal properties.** Construction 5.7 enjoys a number of fairly obvious properties whose proofs are conceptually easy, but lengthy to write down. To keep the size of this already long paper within reason, we leave the proofs of the following facts to the reader.

Fact 5.11 (Compatibility with Kähler differentials). *In Setting 5.2, let $\sigma \in H^0\big(\widehat{X}, \operatorname{Sym}^n \Omega^p_{\widehat{X}}\big)$ be a Kähler tensor, with associated reflexive tensor $\sigma_r \in H^0\big(\widehat{X}, \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big)$. If σ_r is adapted, then the composed morphism*

$$\begin{aligned}H^0\big(\widehat{X}, \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,\gamma)}\big) &\xrightarrow{\varphi^*} H^0\big(Y, \varphi^* \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,\gamma)}\big)\\ &\xrightarrow{H^0(d_C\varphi)} H^0\big(Y, \operatorname{Sym}^n \Omega^p_Y(\log D_Y)\big)\end{aligned}$$

maps the adapted reflexive tensor σ_r to

$$(d\varphi)(\sigma) \in H^0\big(Y, \operatorname{Sym}^n \Omega^p_Y\big) \subseteq H^0\big(Y, \operatorname{Sym}^n \Omega^p_Y(\log D_Y)\big).$$ □

To avoid any potential confusion, we recall that the assumption "σ_r adapted" in Fact 5.11 is equivalent to the assumption that σ_r is contained in the subspace of adapted reflexive tensors,

$$\sigma_r \in H^0\big(\widehat{X}, \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,\gamma)}\big) \subseteq H^0\big(\widehat{X}, \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big).$$

The term $(d\varphi)(\sigma)$ is the classic pull-back of Kähler tensors.

Fact 5.12 (Functoriality). *Let (X, D_X) be a locally uniformizable C-pair and let $(Y_\bullet, D_{Y_\bullet})$ be nc log pairs. Assume that a commutative diagram of the following form is given,*

$$\begin{tikzcd} Y_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha"'] & \widehat{X}_1 \arrow[r, "\gamma_1\text{, } q\text{-morphism}"] \arrow[d, "\beta"] & X \arrow[d, equal] \\ Y_2 \arrow[r, "\varphi_2"'] & \widehat{X}_2 \arrow[r, "\gamma_2\text{, } q\text{-morphism}"'] & X, \end{tikzcd}$$

where

$$\operatorname{supp} \varphi_\bullet^* \gamma_\bullet^* \lfloor D_X \rfloor \subseteq \operatorname{supp} D_{Y_\bullet} \quad \textit{and} \quad \operatorname{supp} \alpha^* D_{Y_2} \subseteq \operatorname{supp} D_{Y_1}.$$

Then, the pull-back morphisms form a commutative diagram of sheaves on Y_1, as follows

$$\begin{tikzcd}
\varphi_1^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma_1)} \arrow[rr, "d_C\varphi_1"] & & \operatorname{Sym}^\bullet \Omega^\bullet_{Y_1}(\log D_{Y_1}) \\
\varphi_1^* \beta^{[*]} \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma_2)} \arrow[u, "\varphi_1^*(\textit{Obs. } 4.14)"] & & \alpha^* \operatorname{Sym}^\bullet \Omega^\bullet_{Y_2}(\log D_{Y_2}) \arrow[u, "d\alpha"'] \\
\varphi_1^* \beta^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma_2)} \arrow[u, "\textit{natl.}"] \arrow[rr, equal] & & \alpha^* \varphi_2^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma_2)} \arrow[u, "\alpha^*(d_C\varphi_2)"']
\end{tikzcd}$$

□

Fact 5.13 (Open immersions). *In Setting 5.2, assume that the morphism $\varphi : Y \to \widehat{X}$ is an open immersion, so we may view Y as an open subset of $\widehat{X}$. The pull-back morphisms $d_C\varphi$ are then equal to the composition of the following sequence of sheaf morphisms,*

$$\begin{aligned}
\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,\gamma)}\big|_Y &\hookrightarrow \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_{\widehat{X}}(\log \gamma^* \lfloor D_X \rfloor)\big|_Y && \textit{Observation } 4.8 \\
&\hookrightarrow \operatorname{Sym}^{[\bullet]} \Omega^{[\bullet]}_Y(\log D_Y) = \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y),
\end{aligned}$$

where the last inclusion is induced by the assumption that $\operatorname{supp} \gamma^* \lfloor D_X \rfloor \subseteq \operatorname{supp} D_Y$. □

Fact 5.14 (Standard operations). *The pull-back morphisms $d_C\varphi$ of Fact 5.9 commute with (reflexive) wedge products, symmetric products and exterior derivatives.* □

5.5.1. *Consequences of the universal properties.* We highlight a few cases that will later become relevant. The following propositions are direct consequences of the compatibility between d_C and the pull-back of Kähler tensors, as stated in Fact 5.11 on the previous page.

Proposition 5.15 (Smooth base spaces). *In Setting 5.2, assume that the space X is smooth and $D_X = 0$, so that $\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,0,\gamma)} = \gamma^* \operatorname{Sym}^\bullet \Omega^\bullet_X \subseteq \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}$ is a sheaf of Kähler tensors. If (Y, D_Y) is nc, then the pull-back morphisms $d_C\varphi$ equal the standard pull-back of Kähler tensors. More precisely, there exist commutative diagrams as follows,*

$$\begin{tikzcd}
\varphi^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,0,\gamma)} \arrow[rr, "d_C\varphi"] \arrow[d, equal] & & \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y) \arrow[d, equal] \\
\varphi^* \gamma^* \operatorname{Sym}^\bullet \Omega^\bullet_X \arrow[r, "d(\gamma\circ\varphi)"'] & \operatorname{Sym}^\bullet \Omega^\bullet_Y \arrow[r, hook] & \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y).
\end{tikzcd}$$

□

Proposition 5.16 (Uniformizations). *In Setting 5.2, assume that the cover γ is a uniformization, so that $\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,0,\gamma)} = \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor)$ is a sheaf of logarithmic Kähler tensors. If (Y, D_Y) is nc, then the pull-back morphisms $d_C\varphi$ equal the standard pull-back of Kähler differentials. More precisely, there exist commutative diagrams as follows,*

$$\begin{tikzcd}
\varphi^* \operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,0,\gamma)} \arrow[r, "d_C\varphi"] \arrow[d, equal] & \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y) \arrow[d, equal] \\
\operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}}(\log \gamma^* \lfloor D \rfloor) \arrow[r, "d\varphi"'] & \operatorname{Sym}^\bullet \Omega^\bullet_Y(\log D_Y).
\end{tikzcd}$$

□

Proposition 5.17 (Constant morphism). *In Setting 5.2, assume that φ is constant. Then, the pull-back morphisms $d_C\varphi$ are zero.*

Proof. Write $\{x\} := \operatorname{img} \varphi$. Applying Fact 5.12 to the diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\text{const}"'] & \widehat{X} \arrow[r, "{\gamma,\ q\text{-morphism}}"] \arrow[d, equal] & X \arrow[d, equal] \\ \{x\} \arrow[r, hook] & \widehat{X} \arrow[r, "\gamma"'] & X, \end{tikzcd}$$

we obtain identifications $d_C\varphi = d\,\mathrm{const} = 0$. □

5.5.2. *Uniqueness.* We mentioned in Section 5.2 that the pull-back morphisms d_C are uniquely determined by "functoriality" and "compatibility with pull-back of Kähler differentials". While true, this might require some explanation. To begin, observe that compatibility with the pull-back of Kähler differentials alone does *not* determine the pull-back morphism $d_C\varphi$ in all possible settings — with the notation of Setting 5.2, think of a case where the image of φ is entirely contained in the singular locus of $\widehat{X}$.

It is however true that compatibility with the pull-back of Kähler differentials and functoriality *together* determine the collection of pull-back morphisms,

$$\big(d_C\varphi\big)_{(\varphi \text{ from a sequence of morphisms as in Setting 5.2})}.$$

Precise statements and proofs are not hard to give, but will be lengthy and painful to spell out. Rather than going into too much detail here, we refer the reader to [Keb13, Sect. 6.4] and [KS21, Sect. 14] that discuss completely analogous situations.

6. Invariants of C-pairs

Almost every invariant defined for compact Kähler manifolds (or logarithmic Kähler pairs) has an analogue in the setting of C-pairs. This section introduces two invariants of particular importance: irregularities and Kodaira-Iitaka dimensions for sheaves of tensors.

6.1. **Irregularities.** For C-pairs, adapted differentials take the role that ordinary differentials play for ordinary spaces. Accordingly, there exists a meaningful notion of "irregularity" for C-pairs. It goes without saying that the irregularity is of fundamental importance when we discuss C-analogues of the Albanese in the follow-up paper [KR24a].

Definition 6.1 (Irregularity, augmented irregularity)**.** *Let (X, D) be a compact C-pair and let $\gamma : \widehat{X} \twoheadrightarrow X$ be any cover. We refer to the number*

$$q(X, D, \gamma) := h^0\Big(\widehat{X},\ \Omega^{[1]}_{(X,D,\gamma)}\Big)$$

as the irregularity of (X, D, γ). *The number*

$$q^+(X, D) := \sup\big\{q(X, D, \gamma) \mid \gamma \text{ a cover}\big\} \in \mathbb{N} \cup \{\infty\}$$

is called augmented irregularity of the C-pair (X, D).

We do not fully understand how the irregularities $q(X, D, \gamma)$ depend on the covering map γ. Section 15.2 gathers several open questions there. Before turning to a C-analogue of the Kodaira-Iitaka dimension in the next subsection, we highlight a few cases where the irregularities can be computed.

Lemma 6.2 (Projective manifolds of small codimension)**.** *Let X be a projective manifold. If X admits an embedding $X \subseteq \mathbb{P}^n$ with $n < 2 \cdot \dim X$, then $q^+(X, 0) = 0$.*

Proof. If $q^+(X, 0) > 0$, then there exists a Galois cover $\gamma : \widehat{X} \twoheadrightarrow X$ with group G and a non-trivial section σ in $\Omega^{[1]}_{(X,0,\gamma)} = \gamma^*\Omega^1_X$. The pluri-differential

$$\prod_{g\in G} g^*\sigma \in H^0\big(\widehat{X},\ \operatorname{Sym}^{\#G} \Omega^{[1]}_{(X,0,\gamma)}\big)$$

is G-invariant, not trivial, and by Lemma 4.20 gives a non-trivial section in $\operatorname{Sym}^{\#G} \Omega^1_X$. It has, however, been shown in [Sch92] that $h^0\big(X, \operatorname{Sym}^N \Omega^1_X\big) = 0$ for every number $N \in \mathbb{N}$. □

To construct a more interesting example, recall from [HP19, Thm. 1.5] that every projective, klt variety X with numerically trivial canonical class admits a cover $\widetilde{X} \twoheadrightarrow X$, étale in codimension one, and a decomposition

$$\widetilde{X} \cong A \times \prod_j Y_j \times \prod_k Z_k$$

where A is an Abelian variety, where the Y_j are (possibly singular) Calabi-Yau varieties and the Z_k are (possibly singular) irreducible symplectic varieties. We refer the reader to [GGK19, Sect. 1.4 and Def. 1.3] for a discussion and for the definition of "singular Calabi-Yau" and "singular irreducible symplectic".

Lemma 6.3. *Let Y be singular Calabi-Yau or singular irreducible symplectic. If $\varphi : X \to Y$ is any birational morphism between normal, projective varieties, and if $D \in \operatorname{Div}(X)$ is a φ-exceptional divisor that makes (X, D) a $\mathcal{C}$-pair, then $q^+(X, D) = 0$.*

Proof. Assume that $q^+(X, D) > 0$. As in the proof of Lemma 6.2, we can then construct a pluri-differential form on $X_{\mathrm{reg}} \setminus \operatorname{supp} D$, hence a non-trivial section $\sigma \in H^0\big(Y^+, \operatorname{Sym}^\bullet \Omega^1_{Y^+}\big)$, where

$$Y^+ := Y_{\mathrm{reg}} \setminus \text{indeterminacy locus of } \varphi^{-1}.$$

Since $Y^+ \subseteq Y$ is a big open subset, σ induces a non-trivial reflexive pluri-differential $\sigma' \in H^0\big(Y, \operatorname{Sym}^{[\bullet]} \Omega^{[1]}_Y\big)$. However, it has been shown in [GGK19, Thm. 1.11] that no such reflexive pluri-differential exist if Y is singular Calabi-Yau or singular irreducible symplectic. □

We refer the reader to [BKT13] for more on the relation between existence of pluri-differentials and the geometry of the underlying space.

Example 6.4 (Unbounded Irregularities). Let X be a compact Riemann surface of general type, and let $\gamma : \widehat{X} \to X$ be any étale cover. Then,

$$q(X, 0, \gamma) = h^0\big(\widehat{X}, \Omega^{[1]}_{(X,0,\gamma)}\big) = h^0\big(\widehat{X}, \gamma^* \Omega^1_X\big) = h^0\big(\widehat{X}, \Omega^1_{\widehat{X}}\big) = g(\widehat{X}).$$

Given that étale covers of arbitrarily high degrees exist, we find that $q^+(X, 0) = \infty$.

6.2. **The $\mathcal{C}$-Kodaira-Iitaka dimension.** This section introduces the $\mathcal{C}$-Kodaira-Iitaka dimension for rank-one sheaves of adapted tensors. We refer the reader to [JK11, Sect. 4] for a related construction, for references, and proper attributions.

Definition 6.5 ($\mathcal{C}$-product sheaves)**.** *Let (X, D) be a $\mathcal{C}$-pair and let $\gamma : \widehat{X} \to X$ be a q-morphism. Assume we are given numbers $n, d, p \in \mathbb{N}^+$ and a coherent subsheaf of adapted reflexive tensors,*

$$\mathscr{F} \subseteq \operatorname{Sym}^{[n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)}.$$

Using the inclusion

$$\operatorname{Sym}^{[d]} \mathscr{F} \subseteq \operatorname{Sym}^{[d]} \operatorname{Sym}^{[n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)} \overset{\text{Obs. 4.11}}{\subseteq} \operatorname{Sym}^{[d\cdot n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)},$$

we define the d^{th} $\mathcal{C}$-product sheaf of $\mathscr{F}$ as

$$\operatorname{Sym}^{[d]}_{\mathcal{C}} \mathscr{F} := \textit{saturation of } \operatorname{Sym}^{[d]} \mathscr{F} \textit{ in } \operatorname{Sym}^{[d\cdot n]}_{\mathcal{C}} \Omega^{[p]}_{(X,D,\gamma)}.$$

Remark 6.6 (Elementary properties). As the saturation of a coherent sheaf within a reflexive sheaf, the product sheaf $\operatorname{Sym}^{[d]}_{\mathcal{C}} \mathscr{F}$ of Definition 6.5 is always reflexive. If $\mathscr{F}$ has rank one, then $\operatorname{Sym}^{[d]}_{\mathcal{C}} \mathscr{F}$ also has rank one.

Definition 6.7 (C-Kodaira-Iitaka dimension). *Let (X,D) be a compact C-pair and let $\gamma : \widehat{X} \twoheadrightarrow X$ be a cover. Given numbers $n, p \in \mathbb{N}^+$ and a coherent, rank-one subsheaf of adapted reflexive tensors,*

$$\mathscr{F} \subseteq \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)},$$

consider the set

$$M := \left\{ m \in \mathbb{N} \,\middle|\, h^0\big(\widehat{X}, \operatorname{Sym}^{[m]}_C \mathscr{F}\big) > 0 \right\}.$$

If $M = \emptyset$, we say that the sheaf $\mathscr{F}$ has C-Kodaira-Iitaka dimension minus infinity *and write $\kappa_C(\mathscr{F}) = -\infty$. Otherwise, consider the natural meromorphic maps*

$$\varphi_m : \widehat{X} \dashrightarrow \mathbb{P}\left(H^0\Big(\widehat{X}, \operatorname{Sym}^{[m]}_C \mathscr{F}\Big)^\vee\right), \text{ for each } m \in M$$

and define the C-Kodaira-Iitaka dimension *as*

$$\kappa_C(\mathscr{F}) = \max_{m \in M} \left\{ \dim \overline{\varphi_m(\widehat{X})} \right\},$$

where $\overline{\varphi_m(\widehat{X})}$ denotes the Zariski closure of $\varphi_m(\widehat{X}) \subseteq \mathbb{P}^\bullet$.

Remark 6.8. Recall from Reminder 2.8 on page 5 that the φ_m are meromorphic indeed, so that $\varphi_m(X) \subseteq \mathbb{P}^\bullet$ are constructible. This implies that the max in the definition of $\kappa_C(\mathscr{F})$ is a maximum and that $\kappa_C(\mathscr{F}) \le \dim X$.

Warning 6.9. Unlike the standard Kodaira-Iitaka dimension, the C-Kodaira-Iitaka dimension is defined only for subsheaves of adapted reflexive differentials. Its value is generally not an invariant of the sheaf alone, and will often depend on the embedding into $\operatorname{Sym}^{[\bullet]}_C \Omega^{[\bullet]}_{(X,D,\gamma)}$.

6.3. **Bogomolov-Sommese vanishing and special pairs.** Campana has observed in [Cam11, Sect. 3.5] that the classic vanishing theorem of Bogomolov and Sommese, [EV92, Cor. 6.9] carries over to C-pairs with simple normal crossing boundary. Using extension theorems for differential forms on log canonical spaces, Patrick Graf generalized Campana's observation substantially in his thesis [Gra13]. While Graf works with projective varieties, his arguments carry over to the setting of compact Kähler spaces[8].

Theorem 6.10 (Bogomolov-Sommese vanishing on X, [Gra15, Thm. 1.2]). *Let (X,D) be a log canonical C-pair where X is compact Kähler. If $\mathscr{F} \subseteq \Omega^{[p]}_{(X,D,\mathrm{Id}_X)}$ is coherent of rank one, then $\kappa_C(\mathscr{F}) \le p$.* □

We say that a pair is special if the inequality in the Bogomolov-Sommese vanishing theorem is strict.

Definition 6.11 (Bogomolov sheaf on X, special pair). *Let (X,D) be a log canonical C-pair where X is compact Kähler. A* Bogomolov sheaf on X *is a coherent sheaf $\mathscr{F} \subseteq \Omega^{[p]}_{(X,D,\mathrm{Id}_X)}$ of rank one such that if $\kappa_C(\mathscr{F}) = p$.*

Definition 6.12 (Special C-pair). *Let (X,D) be a log canonical C-pair where X is compact Kähler. The pair (X,D) is called* special *when there are no Bogomolov sheaves.*

Warning 6.13. We remark that Definition 6.11 differs from Campana's. In [Cam11, Déf. 5.17], Campana defines "specialness" in terms of meromorphic fibrations $X \dashrightarrow Y$ onto orbifolds of general type. For pairs with snc boundary, he shows in [Cam11, Cor. 3.13] that the two definitions agree.

[8] Graf's thesis relies on the paper [GKKP11], which provides the relevant extension theorems for differential forms in the algebraic setting. The newer paper [KS21] establishes analogous results in the analytic setting.

Remark 6.14. The importance of special pairs comes from the existence of the *core map* constructed by Campana [Cam11, Théo. 10.1] for smooth C-pairs: Given a smooth C-pair (X, D) where X is compact Kähler, there exists a unique fibration $c_{(X,D)} : (X, D) \dashrightarrow C(X, D)$ with special generic fibres and orbifold base of general type. Campana's construction therefore splits any smooth C-pair into two antithetic parts, of "special" and "general" type.

Under assumptions that are substantially stronger than those of Theorem 6.10, an analogue of the Bogomolov-Sommese vanishing theorem will hold for sheaves of adapted reflexive tensors on arbitrary covers of X. We include a full statement for future reference and refer the reader to Sections 15.2 and 15.4 for questions concerning potential generalizations.

Proposition 6.15 (Bogomolov-Sommese vanishing on covers of X)**.** *Let (X, D) be a locally uniformizable C-pair where X is compact Kähler. Let $\gamma : \widehat{X} \twoheadrightarrow X$ be a cover. If $\mathscr{F} \subseteq \Omega^{[p]}_{(X,D,\gamma)}$ is coherent of rank one, then $\kappa_C(\mathscr{F}) \le p$.*

Proof. For the reader's convenience, we subdivide the proof into relatively independent steps.

Step 1: Setup. Let $\pi : Y \twoheadrightarrow \widehat{X}$ be a strong log resolution of the pair $(\widehat{X}, \gamma^* \lfloor D \rfloor)$ and consider the reduced divisor

$$D_Y := \Big(\pi^* \gamma^* \lfloor D \rfloor\Big)_{\mathrm{red}} \in \operatorname{Div}(Y).$$

The pair (Y, D_Y) is then snc, and Fact 5.9 provides us with pull-back maps

$$d_C \pi : \pi^* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D,\gamma)} \to \operatorname{Sym}^n \Omega^p_Y(\log D_Y).$$

We consider the saturated images of the C-product sheaves $\operatorname{Sym}^{[n]}_C \mathscr{F}$ and write

$$\mathscr{F}^n_Y := \text{saturation of } (d_C\pi)\Big(\pi^* \operatorname{Sym}^{[n]}_C \mathscr{F}\Big) \text{ in } \operatorname{Sym}^n \Omega^p_Y(\log D_Y).$$

There are two things that we can say immediately.

(6.15.1) The sheaves $\mathscr{F}^n_Y$ are reflexive of rank one. Since Y is smooth, this implies that $\mathscr{F}^n_Y$ are invertible, [OSS11, Lem. 1.1.5].

(6.15.2) The compatibility of pull-back and reflexive symmetric products asserts that

$$\big(\mathscr{F}^1_Y\big)^{\otimes n} = \operatorname{Sym}^n \mathscr{F}^1_Y \overset{\text{Fact 5.14}}{\subseteq} \mathscr{F}^n_Y, \quad \text{for every } n \in \mathbb{N}^+.$$

Step 2: Relation between $\mathscr{F}^1_Y$ and $\mathscr{F}^n_Y$. Following ideas of Patrick Graf [Gra13, Gra15] and simplifying some of his arguments, we will show in this step that the inclusions in Item (6.15.2) are in fact equalities,

$$\big(\mathscr{F}^1_Y\big)^{\otimes n} = \mathscr{F}^n_Y, \quad \text{for every } n \in \mathbb{N}^+. \tag{6.15.3}$$

Since the sheaves on both sides of (6.15.3) are locally free, it suffices to show equality on a suitable big open subset of Y. To this end, recall from the construction of the saturation that $\mathscr{F}^1_Y$ appears on the left of an exact sequence of coherent sheaves on Y,

$$0 \to \mathscr{F}^1_Y \to \Omega^p_Y(\log D_Y) \to \mathscr{Q} \to 0,$$

where $\mathscr{Q}$ is torsion free, and hence locally free over a suitable big, open subset $Y^\circ \subseteq Y$, see [OSS11, Cor. on p. 75]. Since short exact sequences of locally free sheaves are locally split, the sheaf $\mathscr{F}^1_Y|_{Y^\circ}$ is locally a direct summand of $\Omega^p_Y(\log D_Y)|_{Y^\circ}$. But then $\big(\mathscr{F}^1_Y\big)^{\otimes n}|_{Y^\circ}$ is locally a direct summand of $\operatorname{Sym}^n \Omega^p_Y(\log D_Y)|_{Y^\circ}$. The subsheaf $\big(\mathscr{F}^1_Y\big)^{\otimes n}|_{Y^\circ} \subseteq \operatorname{Sym}^n \Omega^p_Y(\log D_Y)|_{Y^\circ}$ is therefore saturated, hence equal to $\mathscr{F}^n_Y|_{Y^\circ}$. Equality (6.15.3) thus follows.

Step 3: End of proof. By construction, sections of the C-product sheaves $\operatorname{Sym}^{[n]}_{C} \mathscr{F}$ induce section of $\mathscr{F}^n_Y$,

$$H^0\Big(\widehat{X},\ \operatorname{Sym}^{[m]}_{C} \mathscr{F}\Big) \hookrightarrow H^0\Big(Y,\ \mathscr{F}^n_Y\Big) \overset{(6.15.3)}{=} H^0\Big(Y,\ (\mathscr{F}^1_Y)^{\otimes n}\Big).$$

If $h^0\big(\widehat{X}, \operatorname{Sym}^{[m]}_{C} \mathscr{F}\big) > 0$, then the associated meromorphic mappings are related,

$$\begin{tikzcd}
Y \arrow[r, dashed, "\psi_m"] \arrow[d, two heads, "\pi"'] & \mathbb{P}\Big(H^0\big(X,\ \mathscr{F}^n_Y\big)^{\vee}\Big) \arrow[d, dashed, two heads, "\text{projection}"] \\
\widehat{X} \arrow[r, dashed, "\varphi_m"'] & \mathbb{P}\Big(H^0\big(\widehat{X},\ \operatorname{Sym}^{[m]}_{C} \mathscr{F}\big)^{\vee}\Big),
\end{tikzcd}$$

so that $\dim \overline{\varphi_m(\widehat{X})} \le \dim \overline{\psi_m(Y)}$. In summary, we find that $\kappa_{\mathscr{C}}(\mathscr{F}) \le \kappa(\mathscr{F}^1_Y)$. The classic Bogomolov-Sommese vanishing theorem, [EV92, Cor. 6.9], asserts that $\kappa(\mathscr{F}^1_Y) \le p$. □

6.4. **Conjectures on the geometry of special pairs.** The class of special C-pairs is supposed to generalize rational or elliptic curves, which suggests the following conjectures made by Campana [Cam11, Conj. 13.10, 13.15, 13.23].

Conjecture 6.16. *Let (X, D) be a smooth C-pair where X is compact Kähler.*

(6.16.1) If $\lfloor D \rfloor = 0$ and (X, D) is special, then the orbifold fundamental group $\pi_1(X, D)$ is almost Abelian.

(6.16.2) The orbifold Kobayashi pseudo-distance $d_{(X,D)}$ vanishes identically if and only if (X, D) is special.

(6.16.3) If (X, D) is projective defined over a number field k, then there exists a finite extension $k' \supset k$ such that rational points $(X, D)(k')$ are dense if and only if (X, D) is special.

Campana has shown that a special compact Kähler manifold X has a surjective Albanese map [Cam04, Prop. 5.3] which implies in particular that its classical augmented irregularity (computed with étale covers) satisfies $\widetilde{q}(X) \le \dim X$. We formulate a conjecture generalizing this property to special C-pairs. It will be discussed in a sequel [KR24a] of this article.

Conjecture 6.17. *Let (X, D) be a log canonical C-pair where X is compact Kähler. If (X, D) is special, then the augmented irregularity satisfies $q^+(X, D) \le \dim X$.*

7. Diagrams admitting pull-back

We feel that the sheaves of adapted reflexive differentials, and in particular the C-cotangent sheaf, are the key objects that make Campana's theory useful. Accordingly, we define a morphism of C-pairs as a morphism of varieties that allows pull-back of adapted reflexive differentials, in a manner that is compatible with the standard pull-back of Kähler differentials, and hence with the pull-back maps introduced in Section 5.4 above.

7.1. **Diagrams admitting pull-back.** The requirement that pull-back be "compatible with the standard pull-back of Kähler differentials and with the pull-back maps introduced in Section 5.4" is conceptually straightforward, but gets somewhat technical to write down correctly. To avoid any potential of confusion, we clarify assumptions and notation explicitly, remind of the relevant facts, and formulate the main definition in great detail.

Setting 7.1 (Commutative diagram of q-morphisms). Assume we are given $\mathcal{C}$-pairs (X, D_X) and (Y, D_Y) and a commutative diagram of the following form,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "a,\ q\text{-morphism}"'] & \widehat{Y} \arrow[d, "b,\ q\text{-morphism}"] \\ X \arrow[r, "\varphi"'] & Y. \end{tikzcd} \tag{7.1.1}$$

Following the conventions of Sections 2.3 and 2.5 write

- $X^\circ := X \setminus \operatorname{supp}\lfloor D_X \rfloor$ for the open part of (X, D_X),
- $Y^\circ := Y \setminus \operatorname{supp}\lfloor D_Y \rfloor$ for the open part of (Y, D_Y), and
- $Y^{\mathrm{lu}} \subseteq Y$ for the maximal open subset over which (Y, D_Y) is locally uniformizable.

We assume that

$$\varphi(X^\circ) \subseteq Y^\circ \quad \text{and} \quad \operatorname{img}\varphi \cap Y^{\mathrm{lu}} \neq \emptyset. \tag{7.1.2}$$

Remark 7.2 (q-morphisms and covers). We stress that the morphisms a and b in (7.1.1) need not be surjective. In other words, we ask that a and b are *q-morphisms* (and hence open, not necessarily surjective), but not necessarily adapted *covers* (which would imply surjective).

The discussion of Setting 7.1 involves a number of additional and auxiliary objects, such as pull-back divisors and preimage sets. For the reader's convenience, we use the following notation consistently throughout the present section.

Notation 7.3 (Divisors, open sets and restrictions in Setting 7.1). In Setting 7.1, consider the reduced divisor

$$\widehat{D}_X := \big(a^*\lfloor D_X \rfloor\big)_{\mathrm{red}} \in \operatorname{Div}(\widehat{X}).$$

We consider the preimage set

$$\widehat{Y}^+ := b^{-1}\big(Y^{\mathrm{lu}}\big) \subseteq \widehat{Y} \quad \text{and let} \quad \widehat{X}^+ \subseteq \widehat{\varphi}^{-1}\big(\widehat{Y}^+\big) \subseteq \widehat{X}$$

be the maximal open subset where $(\widehat{X}, \widehat{D}_X)$ is nc. For brevity, denote the restrictions by

$$\big(\widehat{X}^+, \widehat{D}_X^+\big) := \big(\widehat{X}, \widehat{D}_X\big)\big|_{\widehat{X}^+} \quad \text{and} \quad \widehat{\varphi}^+ := \widehat{\varphi}|_{\widehat{X}^+} : \widehat{X}^+ \to \widehat{Y}^+.$$

The following diagram summarizes the situation,

$$\begin{tikzcd} \widehat{X}^+ \arrow[d, "a|_{\widehat{X}^+}"'] \arrow[rrrr, bend left, "\widehat{\varphi}^+"] & \subseteq & \widehat{X} \arrow[r, "\widehat{\varphi}"'] \arrow[d, "a"'] & \widehat{Y} \arrow[d, "b"] & \supseteq & \widehat{Y}^+ \arrow[d, "b|_{\widehat{Y}^+}"] \\ \varphi^{-1}\big(Y^{\mathrm{lu}}\big) & \subseteq & X \arrow[r, "\varphi"'] & Y & \supseteq & Y^{\mathrm{lu}}. \end{tikzcd}$$

Observation 7.4 (Pull-back morphisms in Setting 7.1). Setting 7.1 reproduces the setup discussed in Section 5, where we introduced the pull-back map for adapted reflexive differentials. Using Notation 7.3, observe that conditions (7.1.2) of Setting 7.1 guarantee that

$$\operatorname{supp}\big(\widehat{\varphi}^+\big)^* b^* \lfloor D_Y \rfloor \subseteq \operatorname{supp} \widehat{D}_X^+ \quad \text{and} \quad \widehat{X}^+ \neq \emptyset.$$

Modulo some differences in the notation, we are therefore in the setting spelled out in 5.2 on page 25. Fact 5.9 will therefore apply to give canonical pull-back morphisms

$$d_{\mathcal{C}}\widehat{\varphi}^+ : \big(\widehat{\varphi}^+\big)^* \operatorname{Sym}_{\mathcal{C}}^{[n]} \Omega^{[\bullet]}_{(Y, D_Y, b)} \to \operatorname{Sym}^n \Omega^\bullet_{\widehat{X}^+}\big(\log \widehat{D}_X^+\big) \tag{7.4.1}$$

that enjoy all properties spelled out in Fact 5.11–5.14 above.

Remark 7.5 (Pull-back and adapted reflexive differentials). For the upcoming discussion, recall from Observation 4.8 that the targets of the pull-back morphisms (7.4.1) contain the sheaves of adapted reflexive tensors,

$$\operatorname{Sym}_C^{[\bullet]} \Omega^{[\bullet]}_{(X,D_X,a)}\big|_{\widehat{X}^+} \subseteq \operatorname{Sym}^\bullet \Omega^\bullet_{\widehat{X}^+}(\log \widehat{D}^+_X).$$

Definition 7.6 (Diagrams admitting pull-back of adapted tensors)**.** *Assume Setting 7.1 and use Notation 7.3. Given numbers $n, p \in \mathbb{N}^+$, we say that $\widehat{\varphi}$* admits pull-back of adapted reflexive (n,p)-tensors *or* Diagram (7.1.1) admits pull-back of adapted reflexive (n,p)-tensors *if there exists a sheaf morphism*

$$\eta : \widehat{\varphi}^* \Big(\operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(Y,D_Y,b)} \Big) \to \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,a)}$$

whose restriction to $\widehat{X}^+$ agrees with the pull-back morphism $d_C\widehat{\varphi}^+$. In other words, there exists a factorization of $d_C\widehat{\varphi}^+$ as follows,

$$\begin{tikzcd} \widehat{\varphi}^*\big(\operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(Y,D_Y,b)}\big)\big|_{\widehat{X}^+} \arrow[r, "\eta|_{\widehat{X}^+}"'] \arrow[rr, bend left, "d_C\widehat{\varphi}^+"] & \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,a)}\big|_{\widehat{X}^+} \arrow[r, hook, "\text{Obs. 4.8}"'] & \operatorname{Sym}^n \Omega^p_{\widehat{X}^+}(\log \widehat{D}^+_X), \end{tikzcd}$$

Notation 7.7 (Diagrams admitting pull-back of adapted differentials). Assume the setting of Definition 7.6. We say that *Diagram* (7.1.1) *admits pull-back of adapted reflexive differentials* if it admits pull-back of adapted reflexive $(1,p)$-tensors, for every $p \in \mathbb{N}^+$.

Warning 7.8 (Not enough to consider $p = 1$). Assume the setup of Definition 7.6. When proving that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials, one might be tempted to hope that it suffices to check that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials only for $p = 1$. This is not the case in general. The sheaf $\Omega^{[1]}_{(Y,D_Y,b)}$ is typically *not* locally free, the natural morphisms

$$\bigwedge^p \Omega^{[1]}_{(Y,D_Y,b)} \to \Omega^{[p]}_{(Y,D_Y,b)}$$

are typically *not* surjective, and sections in $\Omega^{[p]}_{(Y,D_Y,b)}$ can typically *not even locally* be written as products of sections in $\Omega^{[1]}_{(Y,D_Y,b)}$. We refer the reader to Example 8.7 on page 38 for a concrete example.

7.2. **Elementary properties.**

The following observations are elementary but useful. We include them for later reference.

Observation 7.9 (Uniqueness). If it exists at all, then the morphism η of Definition 7.6 is unique. □

Observation 7.10 (Compatibility with pull-back of Kähler tensors). In the setting of Definition 7.6, assume that we are given numbers $n, p \in \mathbb{N}^+$ for which the pull-back morphism η exists. Let $\sigma \in H^0\big(\widehat{Y}, \operatorname{Sym}^n \Omega^p_{\widehat{Y}}\big)$ be a Kähler tensor, with pull-back $\tau \in H^0\big(\widehat{X}, \operatorname{Sym}^n \Omega^p_{\widehat{X}}\big)$ and denote the associated reflexive tensors by

$$\sigma_r \in H^0\big(\widehat{Y}, \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{Y}}\big) \quad \text{and} \quad \tau_r \in H^0\big(\widehat{X}, \operatorname{Sym}^{[n]} \Omega^{[p]}_{\widehat{X}}\big).$$

If σ_r is adapted, then Fact 5.11 implies that τ_r is adapted, and that the composed morphism

$$H^0\big(\widehat{Y}, \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(Y,D_Y,b)}\big) \xrightarrow{\widehat{\varphi}^*} H^0\big(\widehat{X}, \varphi^* \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(Y,D_Y,b)}\big)$$
$$\xrightarrow{H^0(\eta)} H^0\big(\widehat{X}, \operatorname{Sym}_C^{[n]} \Omega^{[p]}_{(X,D_X,a)}\big)$$

maps σ_r to τ_r. □

Observation 7.11 (Local nature). In the setting of Definition 7.6, assume that we are given numbers $n, p \in \mathbb{N}^+$. Let $(\widehat{U}_i)_{i \in I}$ and $(\widehat{V}_j)_{j \in J}$ be open coverings of $\widehat{X}$ and $\widehat{Y}$, respectively. Then, the morphism $\widehat{\varphi}$ admits pull-back of adapted reflexive (n,p)-tensors if and only if every restricted morphism

$$\widehat{\varphi}|_{\widehat{U}_i \cap \widehat{\varphi}^{-1}(\widehat{V}_i)} : \widehat{U}_i \cap \widehat{\varphi}^{-1}(\widehat{V}_i) \to \widehat{V}_i$$

admits pull-back of adapted reflexive (n,p)-tensors. □

If the pull-back map η exists for given numbers $n, p \in \mathbb{N}^+$, it can be seen as a section of the subsheaf

$$\mathscr{H}om_{\widehat{X}}\Big(\widehat{\varphi}^* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b)}, \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D_X,a)}\Big),$$

which is reflexive since $\operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D_X,a)}$ is. To give a section of this sheaf over $\widehat{X}$, it is thus equivalent to give a section over any big open subset.

Observation 7.12 (Removing small subsets). In the setting of Definition 7.6, assume that we are given numbers $n, p \in \mathbb{N}^+$. Let $\widehat{U} \subseteq \widehat{X}$ be a big open subset. Then, the morphism $\widehat{\varphi}$ admits pull-back of adapted reflexive (n,p)-tensors if and only if the morphism $\widehat{\varphi}|_{\widehat{U}}$ admits pull-back of adapted reflexive (n,p)-tensors. □

8. Morphisms of C-pairs

As motivated in the introduction, we define a C-morphism as a morphism where every diagram admits pull-back of adapted reflexive differentials. We impose Condition (7.1.2) to ensure that the word "pull-back of adapted reflexive differentials" carries meaning.

Definition 8.1 (Morphisms of C-pairs)**.** *Given C-pairs (X, D_X) and (Y, D_Y) and a morphism $\varphi : X \to Y$ with*

(8.1.1) $$\varphi(X^\circ) \subseteq Y^\circ \quad \textit{and} \quad \operatorname{img} \varphi \cap Y^{\mathrm{lu}} \neq \emptyset,$$

call φ a morphism between C-pairs (X, D_X) and (Y, D_Y) *if every commutative diagram of form* (7.1.1) *admits pull-back of adapted reflexive differentials.*

Remark 8.2 (Notation used in Definition 8.1). Definition 8.1 uses the standard notation where $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$ denote the open parts of (X, D_X) and (Y, D_Y), and $Y^{\mathrm{lu}} \subseteq Y$ is the maximal open subset over which (Y, D_Y) is locally uniformizable.

Notation 8.3 (C-morphisms). In the setting of Definition 8.1, we will often write $\varphi : (X, D_X) \to (Y, D_Y)$ to indicate that a given morphism $\varphi : X \to Y$ is a morphism between the C-pairs (X, D_X) and (Y, D_Y). We use the word *C-morphism* for brevity.

Warning 8.4 (Pull-back differentials vs. pull-back of tensors). Note that Definition 8.1 asks for pull-back of adapted reflexive differentials and *not* for the more general pull-back of adapted reflexive *tensors*. This is a deliberate design decision, reflecting the fact that differentials, rather than tensors, are the objects that carry geometric meaning. Section 13 discusses criteria to guarantee that some C-morphisms do indeed induce pull-back of adapted reflexive *tensors*.

The following criterion is a direct consequence of Observations 7.11 and 7.12.

Observation 8.5 (Local nature and removing small subsets). Given C-pairs (X, D_X) and (Y, D_Y) and a morphism $\varphi : X \to Y$ such that $\varphi(X^\circ) \subseteq Y^\circ$ and $\operatorname{img} \varphi \cap Y^{\mathrm{lu}} \neq \emptyset$, the following conditions are equivalent.

(8.5.1) The morphism φ is a C-morphism.

(8.5.2) There exist open coverings $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ of X and Y, respectively, such that every restricted morphism

$$\varphi|_{U_i\cap\varphi^{-1}(V_j)} : U_i \cap \varphi^{-1}(V_j) \to V_j$$

is a C-morphism between the restricted C-pairs

$$\bigl(U_i \cap \varphi^{-1}(V_j),\, D_X \cap U_i \cap \varphi^{-1}(V_j)\bigr) \quad \text{and} \quad (V_j, D_Y \cap V_j).$$

(8.5.3) There exists a big open subset $U \subset X$ such that the restriction $\varphi|_U$ is a C-morphism $(U, D_X \cap U) \to (Y, D_Y)$. □

8.1. **First example.** To check if a given morphism is a C-morphism, Definition 8.1 requires us in principle to consider all diagrams of form (7.1.1) and to check if they admit pull-back of adapted reflexive differentials, for all numbers p. This can be cumbersome. We will therefore postpone the discussion of interesting examples until useful criteria are established in the subsequent Section 9. For now, we only mention one example, which we work out in detail.

Example 8.6 (Morphisms to a manifold without boundary). If (X, D_X) is any C-pair and $\varphi : X \to Y$ is any morphism to a manifold Y, then φ is a C-morphism between the pairs (X, D_X) and $(Y, 0)$. For a proof, assume that a diagram of form (7.1.1) is given,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "{a,\ q\text{-morphism}}"'] & \widehat{Y} \arrow[d, "{b,\ q\text{-morphism}}"] \\ X \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

We need to show that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials. To begin, observe that

$$a^{[*]}\Omega^\bullet_X = \Omega^{[\bullet]}_{(X,0,a)} \subseteq \Omega^{[\bullet]}_{(X,D_X,a)} \quad \text{while} \quad \Omega^{[\bullet]}_{(Y,0,b)} = b^*\Omega^\bullet_Y.$$

Pull-back of adapted reflexive differentials is therefore a matter of pulling-back Kähler differentials. To make this precise, follow Notation 7.3 so that $\widehat{X}^+$ is the maximal open set of $\widehat{X}$ where $(\widehat{X}, \widehat{D}_X)$ is nc. Proposition 5.15 on page 28 will then identify the pull-back morphisms for adapted reflexive differentials,

$$d_C\widehat{\varphi}^+ : \quad \underbrace{(\widehat{\varphi}^+)^*\Omega^{[\bullet]}_{(Y,0,b)}}_{=\widehat{\varphi}^* b^*\Omega^\bullet_Y|_{\widehat{X}^+} = a^*\varphi^*\Omega^\bullet_Y|_{\widehat{X}^+}} \quad \to \Omega^\bullet_{\widehat{X}^+}(\log \widehat{D}^+_X),$$

with the pull-back map of Kähler differentials,

$$\begin{tikzcd} a^*\varphi^*\Omega^\bullet_Y|_{\widehat{X}^+} \arrow[r, "a^*(d\varphi)"'] \arrow[rrr, bend left, "d_C\widehat{\varphi}^+"] & a^*\Omega^\bullet_X|_{\widehat{X}^+} \arrow[r, hook, "da"'] & \Omega^\bullet_{\widehat{X}^+} \arrow[r, hook] & \Omega^\bullet_{\widehat{X}^+}(\log \widehat{D}^+_X). \end{tikzcd}$$

It follows that

$$\operatorname{img} d_C\widehat{\varphi}^+ \subseteq \operatorname{img} da|_{\widehat{X}^+} = \Omega^{[\bullet]}_{(X,0,a)}|_{\widehat{X}^+} \subseteq \Omega^{[\bullet]}_{(X,D_X,a)}|_{\widehat{X}^+}.$$

It remains to verify that the morphisms $d_C\widehat{\varphi}^+$ extend from $\widehat{X}^+$ to morphisms

$$\eta : \widehat{\varphi}^*\Omega^{[\bullet]}_{(Y,0,b)} \to \Omega^{[\bullet]}_{(X,D_X,a)}$$

that are defined on all of $\widehat{X}$. Extension is however clear, given that $\Omega^{[\bullet]}_{(Y,0,b)}$ is locally free and $\widehat{X}^+ \subset \widehat{X}$ is a big subset.

8.2. **First Counterexamples.** For singular varieties, Definition 8.1 is quite restrictive. The reader might find it surprising that a morphism between varieties does not always give a morphism of C-pairs, even if target and domain are equipped with the empty boundary.

Example 8.7 (Pull-back for 1-differentials, not for 2-differentials). Consider a construction described in [KS21, Appendix B]: Let E be an elliptic curve and let $L \in \operatorname{Pic}(E)$ be ample. Let Y be the affine cone over E with conormal bundle L and let $\varphi : X \to Y$ be the minimal resolution, obtained by blowing up the vertex. Considering the trivial pairs $(X, 0)$ and $(Y, 0)$, a diagram of the form (7.1.1) is then given as

$$\begin{tikzcd} X \arrow[r, two heads, "\widehat{\varphi}=\varphi"] \arrow[d, hook, two heads, "a\text{, identity}"'] & Y \arrow[d, hook, two heads, "b\text{, identity}"] \\ X \arrow[r, two heads, "\varphi\text{, resolution}"'] & Y. \end{tikzcd}$$

It follows from [KS21, Prop. B.2] that $\widehat{\varphi}$ admits pull-back of reflexive 1-differentials, but not of reflexive 2-differentials. In particular, φ does *not* give a morphism between the C-pairs $(X, 0)$ and $(Y, 0)$.

Example 8.8 (Resolution of the A_1-singularity). Let $\widehat{Y} := \mathbb{A}^2$ and let $b : \widehat{Y} \to Y$ be the quotient morphism for the action of the multiplicative group ± 1, so that Y has a unique singular point, which is of A_1 type. We claim that the minimal resolution morphism $\varphi : X \to Y$ is *not* a morphism between the C-pairs $(X, 0)$ and $(Y, 0)$. In order to see this, construct a diagram as in (7.1.1),

$$\begin{tikzcd} \widehat{X} \arrow[r, two heads, "\widehat{\varphi}\text{, blow-up}"] \arrow[d, two heads, "a\text{, quotient}"'] & \mathbb{A}^2 \arrow[d, two heads, "b\text{, quotient}"] & = & \widehat{Y} \\ X \arrow[r, two heads, "\varphi\text{, blow-up}"'] & \mathbb{A}^2 / \pm 1 & = & Y \end{tikzcd} \tag{8.8.1}$$

The morphism a is two-to-one and ramified exactly along the $\widehat{\varphi}$-exceptional curve in $\widehat{X}$. A direct application of the definitions shows

$$\Omega^{[1]}_{(X,0,a)} = a^* \Omega^1_X \subset \Omega^1_{\widehat{X}} \quad \text{while} \quad \Omega^{[1]}_{(Y,0,b)} = \Omega^1_{\widehat{Y}}.$$

Note however that the differential $d\widehat{\varphi} : \widehat{\varphi}^* \Omega^1_{\widehat{Y}} \to \Omega^1_{\widehat{X}}$ does *not* take its image in $a^* \Omega^1_X$.

Remark 8.9 (Kummer K3s). Example 8.8 shows in particular that the contraction morphism $\varphi : X \twoheadrightarrow Y = A/\pm 1$ from a Kummer K3 surface to its associated torus quotient *is not* a morphism between the C-pairs $(X, 0)$ and $(Y, 0)$. This observation will be of critical importance when we discuss the Albanese of a C-pair in the forthcoming paper [KR24a].

We continue this Example 8.8 in Section 10 on page 42, once suitable criteria for C-morphisms have been established. It will turn out that the morphisms of Example 8.8 and Remark 8.9 do induce C-morphisms once the correct, natural multiplicities of the exceptional sets are taken into account.

9. Criteria for C-morphisms

Throughout the present section, we consider Setting 7.1 in the special case where the q-morphisms are *(adapted) covers* and in particular surjective. We formulate our setup precisely and fix notation.

Setting 9.1. Let (X, D_X) and (Y, D_Y) be two C-pairs and assume that there exists a commutative diagram as follows,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, two heads, "a\text{, arbitrary cover}"'] & \widehat{Y} \arrow[d, two heads, "b\text{, adapted cover}"] \\ X \arrow[r, "\varphi"'] & Y, \end{tikzcd}$$

where $\varphi(X^\circ) \subseteq Y^\circ$ and $\operatorname{img}\varphi \cap Y^{\mathrm{lu}} \neq \emptyset$. Use Notation 7.3/Observation 7.4 and consider the canonical pull-back morphisms

$$d_C\widehat{\varphi}^+ : (\widehat{\varphi}^+)^*\Omega^{[\bullet]}_{(Y,D_Y,b)} \to \Omega^\bullet_{\widehat{X}^+}(\log \widehat{D}^+_X) \tag{9.1.1}$$

Setting 9.1 ends here.

If the morphism $\widehat{\varphi}$ of Setting 9.1 admits pull-back of adapted reflexive differentials, one might be tempted to hope that φ is then a C-morphism. This is not the case in general. The following example shows that *one cannot check that a given morphism is a C-morphism by looking at one pair of covers only, even if both covers are adapted.*

Example 9.2 (Not enough to check one pair of adapted morphisms). Let $\varphi : X \to Y$ be the minimal resolution of the A_1-singularity, as discussed in Example 8.8. Take $a := \mathrm{Id}_X$ and $b := \mathrm{Id}_Y$ and note that the identity morphisms are adapted covers for the pairs $(X, 0)$ and $(Y, 0)$. We are thus in Setting 9.1. It is then very obvious that $\Omega^{[\bullet]}_{(X,0,\mathrm{Id}_X)} = \Omega^\bullet_X$ and $\Omega^{[\bullet]}_{(Y,0,\mathrm{Id}_Y)} = \Omega^{[\bullet]}_Y$, and the existence of pull-back morphisms $\pi^*\Omega^{[\bullet]}_Y \to \Omega^\bullet_X$ is precisely the content of the extension theorem for the A_1-singularity, see [GKK10, Thm. 1.1], [GKKP11, Thm. 1.5] or [KS21, Cor. 1.8]. But we have seen in Example 8.8 that φ is not a C-morphism.

In spite of the negative Example 9.2, there do exist relevant settings where a look at *one* adapted cover and *one* value of p suffices to guarantee that a given morphism of varieties is in fact a morphism of C-pairs. The following proposition and its corollary identify two such cases.

Proposition 9.3 (Criterion for C-morphisms). *In Setting 9.1, assume that $\Omega^{[1]}_{(Y,D_Y,b)}$ is locally free and that there exists a sheaf morphism*

$$\eta^1 : \widehat{\varphi}^*\Omega^{[1]}_{(Y,D_Y,b)} \to \Omega^{[1]}_{(X,D_X,a)}$$

whose restriction to $\widehat{X}^+$ agrees with the canonical pull-back morphisms $d_C\widehat{\varphi}^+$. Then, φ is a C-morphism between the pairs (X, D_X) and (Y, D_Y).

Proposition 9.3 will be shown in Section 9.2 below.

Corollary 9.4 (Criterion for C-morphisms). *In Setting 9.1, assume that $\widehat{Y}$ is smooth of dimension two and that there exists a sheaf morphism*

$$\eta^1 : \widehat{\varphi}^*\Omega^{[1]}_{(Y,D_Y,b)} \to \Omega^{[1]}_{(X,D_X,a)}$$

whose restriction to $\widehat{X}^+$ agrees with the canonical pull-back morphisms $d_C\widehat{\varphi}^+$. Then, φ is a C-morphism between the pairs (X, D_X) and (Y, D_Y).

Proof. Recall from [OSS11, Lem. 1.1.10] that the reflexive sheaf $\Omega^{[1]}_{(Y,D_Y,b)}$ is locally free and apply Proposition 9.3. □

9.1. **Elementary criteria for C-morphisms.** The proof of Proposition 9.3 relies on the following lemma. Since the lemma and the subsequent criterion for a morphism of varieties to be a C-morphism will be used several times in the sequel, we found it worth the while to spell out all details.

Lemma 9.5 (Test for pull-back of adapted reflexive differentials)**.** *Assume we are given two C-pairs, (X, D_X) and (Y, D_Y), and a commutative diagram of morphisms between normal, analytic varieties,*

$$
\begin{tikzcd}
\check{X} \arrow[r, "\check{\varphi}"] \arrow[d, two heads, "{\alpha,\ cover}"'] & \check{Y} \arrow[d, two heads, "{\beta,\ cover}"] \\
\widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "{a,\ q\text{-}morphism}"'] & \widehat{Y} \arrow[d, "{b,\ q\text{-}morphism}"] \\
X \arrow[r, "\varphi"'] & Y.
\end{tikzcd}
\tag{9.5.1}
$$

If $\check{\varphi}$ admits pull-back of adapted reflexive differentials, then $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials.

We do not believe that the converse of Lemma 9.5 holds in general. Proposition 9.3 and Corollary 9.4 identify special situations where a converse can be shown to hold.

Proof of Lemma 9.5. Assuming that $\check{\varphi}$ admits pull-back of adapted reflexive differentials, we need to show that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials. To spell things out: assuming we are given sheaf morphisms

$$\check{\eta} : \check{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b\circ\beta)} \to \Omega^{[\bullet]}_{(X, D_X, a\circ\alpha)}$$

whose restrictions to $\check{X}^+$ agree with the pull-back morphisms $d_C \check{\varphi}^+$, we need to construct appropriate morphisms

$$\widehat{\eta} : \widehat{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b)} \to \Omega^{[\bullet]}_{(X, D_X, a)}$$

whose restrictions to $\widehat{X}^+$ agree with the pull-back morphisms $d_C \widehat{\varphi}^+$. In analogy to Construction 5.7, we consider sheaf morphisms on $\check{X}$,

$$
\begin{aligned}
\alpha^* \widehat{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b)} &= \check{\varphi}^* \beta^* \Omega^{[\bullet]}_{(Y, D_Y, b)} && \text{commutativity} \\
&\to \check{\varphi}^* \beta^{[*]} \Omega^{[\bullet]}_{(Y, D_Y, b)} && \text{natural} \\
&\to \check{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b\circ\beta)} && \text{Observation 4.14} \\
&\to \Omega^{[\bullet]}_{(X, D_X, a\circ\alpha)} && \check{\eta},
\end{aligned}
\tag{9.5.2}
$$

and take $\widehat{\eta}$ as the composition

$$\widehat{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b)} \to \alpha_* \alpha^* \widehat{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b)} \qquad \text{natural} \tag{9.5.3}$$

$$\to \alpha_* \Omega^{[\bullet]}_{(X, D_X, a\circ\alpha)} \qquad \alpha_*(9.5.2) \tag{9.5.4}$$

$$\to \Omega^{[\bullet]}_{(X, D_X, a)} \qquad \text{Consequence 4.18} \tag{9.5.5}$$

We leave it to the reader to apply Fact 5.12 ("Functoriality") in order to check that the restrictions of $\widehat{\eta}$ to $\widehat{X}^+$ indeed agree with $d_C \widehat{\varphi}^+$. □

The following criterion is a direct consequence of Lemma 9.5. In a nutshell, it asserts that to check if a given morphism is a C-morphism, it suffices to restrict attention to those diagrams of the form (7.1.1) where the q-morphisms a and b are adapted.

Corollary 9.6 (Elementary criterion for C-morphisms)**.** *Given C-pairs (X, D_X) and (Y, D_Y) and a morphism $\varphi : X \to Y$, assume that $\varphi(X^\circ) \subseteq Y^\circ$ and $\operatorname{img} \varphi \cap Y^{\mathrm{lu}} \neq \emptyset$.*

Also, assume that every commutative diagram of the form

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "{a,\ adapted}"'] & \widehat{Y} \arrow[d, "{b,\ adapted}"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

admits pull-back of adapted reflexive differentials. Then, φ is a C-morphism between the C-pairs (X, D_X) and (Y, D_Y).

Proof. Observation 8.5 ("being a C-morphism is local on X and Y") and Lemma 2.36 ("strongly adapted covers exist locally") allow assuming without loss of generality that X and Y admit strongly adapted covers. As a consequence, we find that every q-morphism to X and Y can be refined to an adapted morphism via an elementary fibre product construction. More precisely, every commutative diagram of form (7.1.1) can be extended to a diagram of the form (9.5.1), with the additional property that $\alpha \circ a$ and $\beta \circ b$ are adapted for (X, D_X) and (Y, D_Y), respectively. Lemma 9.5 asserts that to prove that $\widehat{\varphi}$ admits pull-back of adapted differentials, it suffices to show that $\check{\varphi}$ admits pull-back of adapted differentials. That, however, holds by assumption. □

9.2. **Proof of Proposition 9.3.** Given a diagram as in Setting 7.1,

$$\begin{tikzcd} \widehat{X}^\circ \arrow[r, "\widehat{\varphi}^\circ"] \arrow[d, "{a^\circ,\ q\text{-morphism}}"'] & \widehat{Y}^\circ \arrow[d, "{b^\circ,\ q\text{-morphism}}"] \\ X \arrow[r, "\varphi"'] & Y, \end{tikzcd}$$

we need to show that $\widehat{\varphi}^\circ$ admits pull-back of adapted reflexive differentials. With this in mind, consider common covers of $\widehat{Y}^\circ$ and $\widehat{Y}$, and $\widehat{X}^\circ$ and $\widehat{X}$ respectively,

$$\check{Y} := \text{normalisation of a component of } \widehat{Y}^\circ \times_Y \widehat{Y}$$
$$\check{X} := \text{normalisation of a component of } \bigl(\widehat{X}^\circ \times_X \widehat{X}\bigr) \times_{\widehat{Y}^\circ} \check{Y}.$$

The following diagram summarizes the situation,

(9.6.1)
$$\begin{tikzcd} \check{X} \arrow[r, equal] \arrow[d, two heads, "\alpha^\circ"'] & \check{X} \arrow[rr, "\check{\varphi}"] \arrow[d, "\alpha"'] & & \check{Y} \arrow[r, equal] \arrow[d, "\beta"] & \check{Y} \arrow[d, two heads, "\beta^\circ"] \\ \widehat{X}^\circ \arrow[d, "a^\circ"'] \arrow[rrrr, bend right, "\widehat{\varphi}^\circ"] & \widehat{X} \arrow[rr, "\widehat{\varphi}"] \arrow[d, two heads, "a"'] & & \widehat{Y} \arrow[d, two heads, "b"] & \widehat{Y}^\circ \arrow[d, "b^\circ"] \\ X \arrow[r, equal] & X \arrow[rr, "\varphi"'] & & Y \arrow[r, equal] & Y. \end{tikzcd}$$

As before, we use Lemma 9.5 ("Test for pull-back of adapted reflexive differentials") and find that it suffices to show that $\check{\varphi}$ admits pull-back of adapted reflexive differentials. In order to construct the relevant morphisms

$$\check{\eta} : \check{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, \beta\circ b)} \to \Omega^{[\bullet]}_{(X, D_X, \alpha\circ a)},$$

consider the identifications

(9.6.2)
$$\begin{aligned} \check{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b\circ\beta)} &= \check{\varphi}^* \beta^{[*]} \Omega^{[\bullet]}_{(Y, D_Y, b)} && \text{Observation 4.15, } b \text{ adapted} \\ &= \check{\varphi}^* \beta^* \Omega^{[\bullet]}_{(Y, D_Y, b)} && \text{local freeness} \\ &= \alpha^* \widehat{\varphi}^* \Omega^{[\bullet]}_{(Y, D_Y, b)} && \text{commutativity.} \end{aligned}$$

In case $\bullet = 1$, the last sheaf admits morphisms as follows,

$$\begin{aligned}
\alpha^* \widehat{\varphi}^* \Omega^{[1]}_{(Y,D_Y,b)} &\to \alpha^* \Omega^{[1]}_{(X,D_X,a)} && \alpha^* \eta^1 \\
&\to \alpha^{[*]} \Omega^{[1]}_{(X,D_X,a)} && \text{natural} \\
&\to \Omega^{[1]}_{(X,D_X,a\circ\alpha)} && \text{Observation 4.14.}
\end{aligned} \tag{9.6.3}$$

In $\bullet = p$ is arbitrary, take $\check{\eta}$ as the composed morphism

$$\begin{aligned}
\check{\varphi}^* \Omega^{[p]}_{(Y,D_Y,\beta\circ b)} &= \alpha^* \widehat{\varphi}^* \Omega^{[p]}_{(Y,D_Y,b)} && (9.6.2) \\
&= \wedge^p \alpha^* \widehat{\varphi}^* \Omega^{[p]}_{(Y,D_Y,b)} && \text{local freeness} \\
&\to \wedge^p \Omega^{[1]}_{(X,D_X,a\circ\alpha)} && \wedge^p \text{ (9.6.3)} \\
&\to \Omega^{[p]}_{(X,D_X,a\circ\alpha)} && \text{natural}
\end{aligned}$$

As before, we leave it to the reader to verify that the morphisms $\check{\eta}$ agree over $\check{X}^+$ with the canonical pull-back morphisms $d_C \check{\varphi}^+$. □

10. Examples and counterexamples

10.1. **Resolution of the A_1-singularity.** To illustrate the use of Proposition 9.3 ("Criterion for C-morphisms"), we continue our discussion on the resolution of the A_1-singularity.

Example 10.1 (Resolution of the A_1-singularity). One concrete example of Setting 9.1 is given in Diagram (8.8.1) of Example 8.8 on page 38. Continuing the notation of the example, denote the φ-exceptional locus by $E \subsetneq X$ and observe that the two-to-one cover a is adapted for the pair $\left(X, \frac{1}{2} \cdot E\right)$. A direct application of the definitions shows

$$\Omega^{[1]}_{\left(X, \frac{1}{2}\cdot E, a\right)} = \Omega^1_{\widehat{X}} \quad \text{and} \quad \Omega^{[1]}_{(Y,0,b)} = \Omega^1_{\widehat{Y}},$$

so that $\Omega^{[1]}_{(Y,0,b)}$ is locally free and that there exists a sheaf morphism

$$\mathrm{d}\,\widehat{\varphi}^\circ \;:\; \widehat{\varphi}^* \Omega^{[1]}_{(Y,0,b)} \to \Omega^{[1]}_{\left(X, \frac{1}{2}\cdot E, a\right)}$$

that agrees with the standard pull-back of Kähler differentials, namely pull-back of Kähler differentials itself. Proposition 9.3 therefore applies to say that φ yields a morphism of C-pairs,

$$\varphi : \left(X, \tfrac{1}{2} \cdot E\right) \to \left(\mathbb{A}^2 \big/ \pm 1,\ 0\right).$$

Remark 10.2 (Kummer K3s). Example 10.1 shows in particular that the contraction morphism $\varphi : X \to Y := A/\pm 1$ from a Kummer K3 surface to its associated torus quotient induces a C-morphism between $\left(X, \frac{1}{2} \cdot E\right)$ and $(Y, 0)$, if $E \subset X$ is the φ-exceptional locus. Again, this observation will be of critical importance when we construct the Albanese of a C-pair in a forthcoming paper.

10.2. **Inclusion of boundary components.** C-morphisms may take their images inside the boundary divisors of the target space. The following example shows the simplest setting.

Example 10.3 (Inclusion of boundary components). Consider a snc C-pair (X, D) where the Weil $\mathbb{Q}$-divisor D is of the form

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i, \quad \text{all } m_i \in \mathbb{N}^{\geq 2}.$$

Pick one component D_0 and define

$$D_0^c := \sum_{i \neq 0} \frac{m_i - 1}{m_i} \cdot D_i|_{D_0} \in \mathbb{Q}\,\mathrm{Div}(D_0).$$

The C-pair (D_0, D_0^c) is then snc, and we claim that the inclusion $\iota : D_0 \to X$ is a C-morphism between the pairs (D_0, D_0^c) and (X, D). Since the claim is local, we may assume without loss of generality that $X = \mathbb{B} \subset \mathbb{C}^n$ is the unit ball, that $\operatorname{supp} D \subsetneq X$ is a union of hyperplanes, and that we are given a uniformization

$$\gamma : \widehat{X} \to X, \quad (x_1, x_2, \dots, x_n) \mapsto \big(x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}\big),$$

where $\widehat{X} := \mathbb{B}$ is again the unit ball. Consider the preimage $\widehat{D}_0 := \gamma^{-1}(D_0) \subsetneq \widehat{X}$ with its reduced structure and observe that $\gamma|_{\widehat{D}_0} : \widehat{D}_0 \to D_0$ is again a uniformization. We obtain a diagram

$$\begin{tikzcd} \widehat{D}_0 \arrow[r, hook, "\iota\text{,inclusion}"] \arrow[d, two heads, "\gamma|_{\widehat{D}_0}\text{, uniformization}"'] & \widehat{X} \arrow[d, two heads, "\gamma\text{, uniformization}"] \\ D_0 \arrow[r, hook, "\text{inclusion}"'] & X. \end{tikzcd}$$

Observing that

$$\Omega^{[\bullet]}_{(D_0, D_0^c, \gamma|_{\widehat{D}_0})} = \Omega^{\bullet}_{\widehat{D}_0} \quad \text{and} \quad \Omega^{[\bullet]}_{(X,D,\gamma)} = \Omega^{\bullet}_{\widehat{X}}$$

and that $d_C\iota : \iota^*\Omega^{[\bullet]}_{(X,D,\gamma)} \to \Omega^{\bullet}_{\widehat{D}_0}$ is the restriction of Kähler differentials, Proposition 9.3 yields the claim.

10.3. **Comparison of divisors.** We continue with a criterion for the identity morphism to be a C-morphism under a change of divisors. While perhaps trivial, the criterion is so useful that it deserves to be mentioned and carefully proven.

Proposition 10.4 (Comparison of divisors). *Let $(X, D_{1,X})$ and $(X, D_{2,X})$ be two C-pairs on the same underlying space. Then, the following statements are equivalent.*

(10.4.1) The identity morphism Id_X is a C-morphism between $(X, D_{1,X})$ and $(X, D_{2,X})$.
(10.4.2) We have $D_{1,X} \geq D_{2,X}$.

Proof of Proposition 10.4, (10.4.1) ⇒ (10.4.2). If $x \in X$ is any point where $\big(X, D_{1,X} + D_{2,X}\big)$ is nc, there exists a q-morphism $\gamma : \widehat{X} \to X$ where $\big(\widehat{X}, \gamma^*(D_{1,X} + D_{2,X})\big)$ is nc, where γ is adapted for $(X, D_{1,X})$ and for $(X, D_{2,X})$, and contains x in its image. We obtain a diagram as follows,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\mathrm{Id}_{\widehat{X}}"] \arrow[d, "\gamma\text{, adapted } q\text{-morphism}"'] & \widehat{X} \arrow[d, "\gamma\text{, adapted } q\text{-morphism}"] \\ X \arrow[r, "\mathrm{Id}_X"'] & X. \end{tikzcd}$$

The morphism $\mathrm{Id}_{\widehat{X}}$ admits pull-back of adapted reflexive differentials by assumption. In other words, there exists an inclusion of sheaves, $\Omega^1_{(X,D_{2,X},\gamma)} \subseteq \Omega^1_{(X,D_{1,X},\gamma)}$. The definition of $\Omega^1_{(X,D_{\bullet,X},\gamma)}$ will then imply that the inequality $D_{1,X} \geq D_{2,X}$ holds over the open set $\mathrm{img}(\gamma)$. The claim follows since x is an arbitrary point in a big open subset of X. □

Proof of Proposition 10.4, (10.4.2) ⇒ (10.4.1). Given a diagram as follows

$$\begin{tikzcd} \widehat{X}_1 \arrow[r, "\widehat{\varphi}"] \arrow[d, "\gamma_1\text{, } q\text{-morphism}"'] & \widehat{X}_2 \arrow[d, "\gamma_2\text{, } q\text{-morphism}"] \\ X \arrow[r, "\mathrm{Id}_X"'] & X, \end{tikzcd}$$

we need to show that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials. A few simplifications can be made without loss of generality. To begin, observe that $\widehat{\varphi}$ is q-morphism and hence open by Reminder 2.19 on page 6. Replacing $\widehat{X}_2$ with the image of $\widehat{\varphi}$, we may therefore assume without loss of generality that $\widehat{\varphi}$ is surjective. We can then invoke Lemma 9.5, replace γ_2 by $\gamma_2 \circ \widehat{\varphi}$ and assume without loss of generality that γ_1 and γ_2 are equal, and that $\widehat{\varphi}$ is the identity map on $\widehat{X}_1 = \widehat{X}_2$.

With these simplifications in place, it is clear that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials if and only if we have inclusions

$$\Omega^{[p]}_{(X,D_{2,X},\gamma)} \subseteq \Omega^{[p]}_{(X,D_{1,X},\gamma)} \quad \text{for every } p.$$

The inequality $D_{1,X} \geq D_{2,X}$ will however guarantee that. □

10.4. **$\mathcal{C}$-Resolutions of singularities.** Regretfully, it follows almost immediately from the definition of $\mathcal{C}$-morphism that resolutions of singularities do not always exist.

Definition 10.5 ($\mathcal{C}$-Resolution of singularities)**.** *Let (X, D) be a $\mathcal{C}$-pair. A $\mathcal{C}$-*resolution of singularities *is a proper, bimeromorphic $\mathcal{C}$-morphism $\pi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ where $(\widetilde{X}, \widetilde{D})$ is snc and $\pi_* \widetilde{D} = D$.*

Proposition 10.6 (Necessary criterion for existence $\mathcal{C}$-resolutions of singularities)**.** *Let X be a normal analytic variety. Assume that X is Gorenstein and that a $\mathcal{C}$-resolution of singularities $\pi : (\widetilde{X}, \widetilde{D}) \to (X, 0)$ exists. Then, X is log canonical. In particular, X has Du Bois singularities.*

Once appropriate criteria are established, we generalize Proposition 10.6 in Corollary 13.5 to locally $\mathbb{Q}$-Gorenstein $\mathcal{C}$-pairs with non-trivial boundary.

Proof of Proposition 10.6. Let $E \in \operatorname{Div}(\widetilde{X})$ denote the π-exceptional divisor, with its reduced structure. Consider the trivial diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[d, "\mathrm{Id}_{\widetilde{X}}\text{, } q\text{-morphism}"'] & X \arrow[d, "\mathrm{Id}_X\text{, } q\text{-morphism}"] \\ \widetilde{X} \arrow[r, "\pi"'] & X \end{tikzcd}$$

Definition 8.1 guarantees that φ admits pull-back of adapted reflexive differentials,

$$\begin{aligned} \pi^*\omega_X = \pi^*\left(\Omega^{[\dim X]}_{(X,0,\mathrm{Id}_X)}\right) & & \text{Example 4.6} \\ \to \Omega^{[\dim X]}_{(\widetilde{X},\widetilde{D},\mathrm{Id}_{\widetilde{X}})} & & \text{pull-back} \\ = \omega_{\widetilde{X}}(\log\lfloor\widetilde{D}\rfloor) \subseteq \omega_{\widetilde{X}}(\log E) = \omega_{\widetilde{X}}(E) & & \text{Example 4.6.} \end{aligned}$$

By definition, [KM98, Sec. 2.3], this means that $\operatorname{discrep}(X, 0) \geq -1$ so that X is log canonical. □

Example 10.7 ($\mathcal{C}$-pair without $\mathcal{C}$-resolution of singularities)*.* For a concrete example of a variety that is Gorenstein but not log canonical, let $Y \subsetneq \mathbb{P}^2$ be any general type curve, and let $X \subset \mathbb{A}^3$ be the affine cone over Y with normal bundle $\mathscr{O}_{\mathbb{P}^2}(1)|_Y$, as discussed in [KS21, App. B]. The variety X is then normal. As a hypersurface in $\mathbb{A}^3$, it is also Gorenstein. However, we have seen in [KS21, Prop. B.3] that X is not log canonical.

For future reference, we note the following variant of Proposition 10.6, which relates the existence of $\mathcal{C}$-resolutions of singularities to the notion of "weakly rational" singularities, as introduced in [KS21, Sect. 1.4 and Def. A.1].

Proposition 10.8 (Necessary criterion for the existence of special C-resolutions of singularities)**.** *Let X be a C-pair with $\lfloor D \rfloor = 0$. Assume that there exists a C-resolution of singularities,*

$$\pi : (\widetilde{X}, \widetilde{D}) \to (X, D),$$

where $\lfloor \widetilde{D} \rfloor = 0$. Then, X has weakly rational singularities in the sense of [KS21, Def. A.1].

Remark 10.9 (Rational and weakly rational singularities)*.* If the space X of Proposition 10.8 is Cohen-Macaulay, recall from [KS21, Sect. 1.4 and references there] that X has weakly rational singularities if and only if it has rational singularities.

Proof of Proposition 10.8. As in the proof of Proposition 10.6, we obtain a pull-back map for adapted reflexive differentials,

$$\pi^* \omega_X = \pi^* \Big(\Omega^{[\dim X]}_{(X, D, \mathrm{Id}_X)} \Big) \to \Omega^{[\dim X]}_{(\widetilde{X}, \widetilde{D}, \mathrm{Id}_{\widetilde{X}})} = \omega_{\widetilde{X}}.$$

In the language of [KS21, Sect. 1.4], this implies that the Grauert-Riemenschneider sheaf on X equals its dualizing sheaf,

$$\omega_X^{\mathrm{GR}} \overset{\text{def.}}{=} \pi_* \omega_{\widetilde{X}} = \omega_X,$$

which is reflexive. By definition, this means that X is weakly rational singularities. □

Proposition 10.10 (Sufficient criterion for existence of C-resolutions of singularities)**.** *Let (X, D) be a locally uniformizable C-pair. Then, a C-resolution of singularities exists.*

Remark 10.11 (Proposition 10.10 is not optimal)*.* Proposition 10.10 is far from optimal. It is certainly possible to bound the coefficients of the resolution pair.

Proof of Proposition 10.10. Let $\pi : Y \to X$ be a strong log resolution of the pair (X, D), with exceptional divisor $E \subset Y$. Consider the divisor $D_Y := \pi_*^{-1} D + E \in \operatorname{Div} Y$, where $\pi_*^{-1} D$ denotes the strict transform. We claim that π is a morphism between the C-pairs (Y, D_Y) and (X, D). Recalling from Observation 8.5 that the claim is local on X, we assume without loss of generality that X is uniformizable, so that a diagram of the following form exists,

$$\begin{tikzcd} \widehat{Y} \arrow[r, "\widehat{\pi}"] \arrow[d, twoheadrightarrow, "\gamma_Y"'] & \widehat{X} \arrow[d, twoheadrightarrow, "{\gamma_X,\ \text{uniformization}}"] \\ Y \arrow[r, "{\pi,\ \text{resolution}}"'] & X, \end{tikzcd} \tag{10.12.1}$$

where $\widehat{Y}$ is obtained as the normalization of a suitable component in the fibre product $Y \times_X \widehat{X}$. Further simplifications are possible: Observation 8.5 allows assuming without loss of generality that Y and $\widehat{Y}$ are smooth, and that the divisors D_Y and $\gamma_Y^* D_Y$ have smooth support.

Since γ_X uniformizes, we have seen in Example 4.6 that $\Omega^{[1]}_{(X, D, \gamma_X)}$ is locally free. Proposition 9.3 therefore applies to show that π is a morphism of C-pairs if and only if Diagram (10.12.1) admits pull-back of adapted reflexive differentials. That, however, follows almost immediately from our choice of a boundary divisor on Y and from Fact 5.9 on page 27. To be precise, recall that Fact 5.9 equips us with a pull-back map

$$\begin{aligned} \widehat{\pi}^* \Omega^1_{\widehat{X}}(\log \gamma_X^* \lfloor D_X \rfloor) &= \widehat{\pi}^* \Omega^{[1]}_{(X, D_X, \gamma_X)} && \gamma_Y \text{ uniformizes} \\ &\to \Omega^1_{\widehat{Y}}(\log \widehat{\pi}^* \gamma_X^* \lfloor D_X \rfloor) && \text{pull-back } d_C \varphi \\ &= \Omega^1_{\widehat{Y}}(\log \gamma_Y^* \pi^* \lfloor D_X \rfloor) && \text{commutativity} \\ &\hookrightarrow \Omega^1_{\widehat{Y}}(\log \gamma_Y^* \lfloor D_Y \rfloor) && \text{choice of } D_Y. \end{aligned} \tag{10.12.2}$$

To prove that Diagram (10.12.1) admits pull-back of adapted reflexive differentials, we need to show that the composed map (10.12.2) takes its image in

$$\Omega^1_{(Y, D_Y, \gamma_Y)} \subseteq \Omega^1_{\widehat{Y}}(\log \gamma_Y^* \lfloor D_Y \rfloor). \tag{10.12.3}$$

This is clear over the complement of E, where π and $\widehat{\pi}$ are isomorphic. This is also clear over the complement of $\pi_*^{-1} D$, where D_Y is reduced. Recalling the assumption that D_Y has smooth support, observe that E and $\pi_*^{-1} D$ are disjoint, so that (10.12.3) holds everywhere. □

11. Functoriality

Observe that C-pairs form no category because they cannot be composed. If

$$(X, D_X) \xrightarrow{\varphi_1} (Y, D_Y) \xrightarrow{\varphi_2} (Z, D_Z) \tag{11.0.1}$$

is a sequence of morphisms of C-pairs, the composition $\varphi_2 \circ \varphi_1$ need not be a C-morphism between (X, D_X) and (Z, D_Z), for the simple reason that the image of the composed morphism might be disjoint from the open set $Z^{\mathrm{lu}} \subseteq Z$ where (Z, D_Z) is locally uniformizable. The following proposition asserts that this is the only obstacle for functoriality. It implies in particular that locally uniformizable C-pairs form a category.

Proposition 11.1 (Weak functoriality)**.** *Given a sequence of morphisms between C-pairs as in* (11.0.1)*, assume that*

$$\operatorname{img}(\varphi_2 \circ \varphi_1) \cap Z^{\mathrm{lu}} \neq \emptyset.$$

Then, $\varphi_2 \circ \varphi_1$ is a morphism between the C-pairs (X, D_X) and (Z, D_Z).

Proof. We apply the elementary criterion for C-morphisms spelled out in Corollary 9.6 above: assuming that we have a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "a\text{, adapted}"'] & \widehat{Z} \arrow[d, "c\text{, adapted}"] \\ X \arrow[r, "\varphi_2 \circ \varphi_1"'] & Z, \end{tikzcd}$$

we need to show that $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials. The question is local on X. We can therefore shrink all relevant spaces and assume that Y is Stein. Lemma 2.36 will then yield an adapted cover $\overline{Y} \twoheadrightarrow Y$. We can thus set

$$\begin{aligned} \check{Y} &:= \text{normalisation of a component of } \overline{Y} \times_Z \widehat{Z}, \\ \check{X} &:= \text{normalisation of a component of } \check{Y} \times_Y \widehat{X}, \end{aligned}$$

and obtain a diagram as follows,

$$\begin{tikzcd} \check{X} \arrow[r, "\check{\varphi}_1"] \arrow[d, two heads, "\alpha"'] & \check{Y} \arrow[r, "\check{\varphi}_2"] \arrow[dd, two heads, "\beta"'] & \check{Z} \arrow[d, equal, "\gamma"] \\ \widehat{X} \arrow[rr, "\widehat{\varphi}"'] \arrow[d, "a"'] & & \widehat{Z} \arrow[d, "c"] \\ X \arrow[r, "\varphi_1"'] & Y \arrow[r, "\varphi_2"'] & Z. \end{tikzcd}$$

Lemma 9.5 applies to this setting, so that we only need to show that $\check{\varphi}_2 \circ \check{\varphi}_1$ admits pull-back of adapted reflexive differentials. But each morphism $\check{\varphi}_\bullet$ admits pull-back of adapted reflexive differentials individually. □

12. Existence of categorical quotients

C-pairs admit a natural notion of a categorical quotient under the action of a finite group. The following definitions are direct analogues of the classic definitions for normal varieties and should not come as a surprise.

Notation 12.1 (Group action on C-pair). Let (X, D_X) be a C-pair and let G be a finite group. A *G-action on* (X, D_X) is a G-action on X that stabilizes the divisor D_X.

Remark 12.2. The requirement that the action stabilizes D_X is equivalent to the requirement that $g^* D_X = D_X$ for all $g \in G$. We do not require that the G-action stabilizes the components of D_X individually, nor that it fixes them pointwise.

Definition 12.3 (Categorical quotients of C-pairs)**.** *Let G be a finite group that acts on a C-pair (X, D_X). A* categorical quotient of (X, D_X) by G *is a surjective morphism of C-pairs,*

$$q : (X, D_X) \twoheadrightarrow (Q, D_Q),$$

whose underlying morphism of varieties is constant on G-orbits and that satisfies the following universal property: if $\varphi : (X, D_X) \to (Y, D_Y)$ is any morphism of C-pairs whose underlying morphism of varieties is constant on G-orbits, then φ factorizes via q,

$$\begin{tikzcd} (X, D_X) \arrow[r, two heads, "q"'] \arrow[rr, bend left, "\varphi"] & (Q, D_Q) \arrow[r, "\exists! \psi"'] & (Y, D_Y). \end{tikzcd}$$

At the level of underlying spaces, the quotient is simply the categorical quotient of a normal analytic space, [Car57, Thm. 4]. The following construction equips the quotient space with a suitable divisor.

Construction 12.4. Let (X, D_X) be a C-pair and let G be a finite group that acts on (X, D_X). Set $Q := X/G$ and take $q : X \to Q$ as the quotient morphism. For any prime Weil divisor $H \in \operatorname{Div}(Q)$, choose a component of the preimage $H' \subset \operatorname{supp} q^* H$ and set

$$m_H := (\operatorname{mult}_{H'} q^* H) \cdot (\operatorname{mult}_{C,H'} D_X).$$

Observe that the number m_H does not depend on the choice of H' and set

$$D_Q := \sum_{H \in \operatorname{Div}(Q) \text{ prime}} \frac{m_H - 1}{m_H} \cdot H \in \mathbb{Q}\operatorname{Div}(Q).$$

As before, we stick to the convention that

$$\infty \cdot (\text{positive, finite}) = \infty, \quad \infty - (\text{finite}) = \infty, \quad \text{and} \quad \infty/\infty = 1.$$

Theorem 12.5 (Existence of quotients)**.** *In the setting of Construction 12.4, the morphism q is a morphism of C-pairs, $q : (X, D_X) \to (Q, D_Q)$, and this morphism is a categorical quotient.*

The proof of Theorem 12.5 is elementary, but the somewhat delicate notion of C-morphism does require some attention and makes the proof a little more technical than we would have preferred. For the reader's convenience, we defer the proof until Subsection 12.4 and discuss a few properties of the quotient construction first.

Notation 12.6. The universal property implies immediately that categorical quotients are unique up to unique isomorphism. We will therefore speak of "the" quotient and denote the quotient C-pair by the symbol $(X, D_X)/G$.

12.1. **Functoriality.** The universal property in the definition of "quotient" will often be used in the following form, which we formulate separately for the reader's convenience.

Proposition 12.7 (Functoriality)**.** *Let G be a finite group that acts on two C-pairs, (X, D_X) and (Y, D_Y). Let $\gamma : (X, D_X) \to (Y, D_Y)$ be a G-equivariant morphism of C-pairs. Then, there exists an induced morphism between categorical quotients, and a commutative diagram of morphisms between C-pairs as follows,*

$$\begin{tikzcd} (X, D_X) \arrow[r, "\gamma"] \arrow[d, two heads, "q_X,\ quotient"'] & (Y, D_Y) \arrow[d, two heads, "q_Y,\ quotient"] \\ (X, D_X) \big/ G \arrow[r, "\gamma^G"'] & (Y, D_Y) \big/ G. \end{tikzcd}$$

Proposition 12.7 is an almost immediate consequence of the following lemma, which we show in Section 12.3 below.

Lemma 12.8 (Quotients of uniformizable pairs)**.** *Quotients of uniformizable pairs are uniformizable. Quotients of locally uniformizable pairs are locally uniformizable.*

Proof of Proposition 12.7. Proposition 11.1 ("Weak functoriality") and Lemma 12.8 imply that $q_Y \circ \gamma$ is a morphism of C-pairs. Since $q_Y \circ \gamma$ is constant on the orbits of the G-action on X, it will factor via q_X. □

12.2. **Quotients as adapted covers.** The following two observations are frequently useful. The elementary proofs are left to the reader.

Observation 12.9 (Adapted covers vs. quotients). Let (X, D_X) be a C-pair and let $\gamma : \widehat{X} \twoheadrightarrow X$ be an adapted cover that is Galois with group G. Write

$$D_{\widehat{X}} := \big(\gamma^* \lfloor D_X \rfloor\big)_{\text{red}} \quad \text{and} \quad (X, D'_X) := (\widehat{X}, D_{\widehat{X}}) \big/ G.$$

Then, $D'_X \geq D_X$. Proposition 10.4 on page 43 allows formulating this inequality by saying that the identity induces a morphism of C-pairs,

$$\mathrm{Id}_X : (X, D'_X) \to (X, D_X). \qquad \square$$

Observation 12.10 (Quotients vs. adapted covers). Let G be a finite group that acts on a log pair (X, D_X) and let $\gamma : X \twoheadrightarrow X/G$ be the quotient morphism. Then, γ is a strongly adapted cover for the quotient C-pair $(Q, D_Q) := (X, D_X)/G$. The C-cotangent sheaf equals $\Omega^{[1]}_{(Q, D_Q, \gamma)} = \Omega^{[1]}_X(\log D_X)$. □

12.3. **Adapted covers, uniformizations and quotients. Proof of Lemma 12.8.** The following lemma is key to the proofs of Theorem 12.5, Proposition 12.7, and Lemma 12.8. It might also be of independent interest.

Lemma 12.11 (Adapted covers and quotients)**.** *In the setting of Construction 12.4, let $\gamma : \widehat{X} \to X$ be a q-morphism. If γ is adapted for the pair (X, D_X), then $q \circ \gamma : \widehat{X} \to Q$ is adapted for the pair (Q, D_Q) and*

$$\Omega^{[p]}_{(X, D_X, \gamma)} = \Omega^{[p]}_{(Q, D_Q, q \circ \gamma)} \quad \textit{for every number } p.$$

Proof. The assertion that $q \circ \gamma$ is adapted follows directly from the choices made in Construction 12.4.

Two reflexive sheaves $\widehat{X}$ agree if they agree on a big open set. Removing codimension-two subsets from all relevant varieties, we can therefore assume without loss of generality that all spaces are smooth and that all divisors have smooth support. We are therefore in the setting of Definition 3.2 ("Bundle of adapted tensors in the nc case") where the relevant C-cotangent sheaves are given as

$$\Omega^1_{(X,D_X,\gamma)} = \ker\Big(\gamma^*\Omega^1_X(\log D_X) \xrightarrow{\gamma^*(\text{residue morphism})} \mathscr{O}_{\gamma^* D_{X,\mathrm{orb}}}\Big). \tag{12.11.1}$$

$$\Omega^1_{(Q,D_Q,q\circ\gamma)} = \ker\Big(\gamma^* q^*\Omega^1_Q(\log D_Q) \xrightarrow{\gamma^* q^*(\text{residue morphism})} \mathscr{O}_{\gamma^* q^* D_{Q,\mathrm{orb}}}\Big). \tag{12.11.2}$$

It suffices to consider the case $p = 1$ and to show equality of these sheaves only; equality for all other values of p will follow by local freeness.

The construction of the divisor D_Q has two immediate consequences. To formulate them properly, let R be the reduced divisor on X obtained as the union of those components of the ramification divisor that are not contained in the finite part of D_X,

$$R := (\operatorname{supp} \operatorname{Ramification} q) \setminus \operatorname{supp} D_{X,\mathrm{orb}} \quad \in \operatorname{Div}(X).$$

With this notation in place, it follows from construction that

$$\operatorname{Branch} q \subseteq \operatorname{supp} D_Q \quad \text{and} \quad q^* D_{Q,\mathrm{orb}} = D_{X,\mathrm{orb}} + R.$$

The inclusion implies in particular that $q^*\Omega^1_Q(\log D_Q) = \Omega^1_X(\log D_X + R)$, and (12.11.2) simplifies to

$$\begin{aligned}
\Omega^1_{(Q,D_Q,q\circ\gamma)} &= \ker\Big(\gamma^*\Omega^1_X(\log D_X + R) \xrightarrow{\gamma^*(\text{residue morphism})} \mathscr{O}_{\gamma^*(D_{X,\mathrm{orb}}+R)}\Big) \\
&= \ker\Big(\gamma^*\Omega^1_X(\log D_X + R) \xrightarrow{\gamma^*(\text{residue morphism})} \mathscr{O}_{\gamma^* D_{X,\mathrm{orb}}} \oplus \mathscr{O}_{\gamma^* R}\Big);
\end{aligned}$$

for the last equality, we use that $\operatorname{supp} D_{X,\mathrm{orb}}$ and $\operatorname{supp} R$ are disjoint by our smoothness assumption. Recalling the standard fact that $\Omega^1_X(\log D_X)$ is described as the kernel of the following residue morphism,

$$\Omega^1_X(\log D_X) = \ker\Big(\Omega^1_X(\log D_X + R) \xrightarrow{(\text{residue morphism})} \mathscr{O}_{\gamma^* R}\Big)$$

the claim now follows. □

Proof of Lemma 12.8. We consider the uniformizable case only. Let (X, D_X) be a C-pair, let G be a finite group that acts on (X, D_X) and let (Q, D_Q) be the quotient of Construction 12.4. Finally, consider a cover $\gamma : \widehat{X} \to X$ where $\big(\widehat{X}, (\gamma^*\lfloor D_X\rfloor)_{\mathrm{reg}}\big)$ is nc. The following statements are then equivalent.

$$\begin{aligned}
\gamma \text{ uniformizes } (X, D_X) &\Leftrightarrow \Omega^{[\bullet]}_{(X,D_X,\gamma)} = \Omega^\bullet_{\widehat{X}}(\log \gamma^*\lfloor D_X\rfloor) && \text{Observation 4.9} \\
&\Leftrightarrow \Omega^{[\bullet]}_{(Q,D_Q,q\circ\gamma)} = \Omega^\bullet_{\widehat{X}}(\log \gamma^*\lfloor D_X\rfloor) && \text{Lemma 12.11} \\
&\Leftrightarrow \Omega^{[\bullet]}_{(Q,D_Q,q\circ\gamma)} = \Omega^\bullet_{\widehat{X}}(\log (q\circ\gamma)^*\lfloor D_Q\rfloor) && \text{Construction 12.4} \\
&\Leftrightarrow q\circ\gamma \text{ uniformizes } (Q, D_Q). && \text{Observation 4.9}
\end{aligned}$$

The claim thus follows. □

12.4. **Existence of quotients, proof of Theorem 12.5.** Maintain setting and assumptions of Theorem 12.5 and Construction 12.4. The theorem asserts that q is a morphism of C-pairs and that the universal property holds. Even though the proofs of these two statements are similar, we prefer to present the arguments separately, in two separated steps.

Step 1: The quotient morphism is a morphism of C-pairs. We work with Corollary 9.6 ("Elementary criterion for C-morphisms") and assume that a commutative diagram of the following form is given,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{q}"] \arrow[d, "\gamma\text{, adapted}"'] & \widehat{Q} \arrow[d, "\delta\text{, adapted}"] \\ X \arrow[r, two heads, "q\text{, quotient morphism}"'] & Q. \end{tikzcd}$$

In order to show that the quotient morphism q is a morphism of C-pairs, we need to show that $\widehat{q}$ admits pull-back of adapted reflexive differentials. That, however, follows now almost directly,

$$\begin{aligned} \Omega^{[\bullet]}_{(X,D_X,\gamma)} &= \Omega^{[\bullet]}_{(Q,D_Q,q\circ\gamma)} && \text{Lemma 12.11} \\ &= \Omega^{[\bullet]}_{(Q,D_Q,\delta\circ\widehat{q})} && \text{commutativity} \\ &= \widehat{q}^{[*]}\Omega^{[\bullet]}_{(Q,D_Q,\delta)} && \text{Observation 4.14.} \end{aligned}$$

Step 2: Universal property. Let $\varphi : (X, D_X) \to (Y, D_Y)$ be any morphism of C-pairs whose underlying morphism of varieties is constant on G-orbits. Then, the universal property of the classic categorical quotients asserts that φ factorizes via q as a morphism of analytic varieties,

$$\begin{tikzcd} X \arrow[r, two heads, "q"'] \arrow[rr, bend left, "\varphi"] & Q \arrow[r, "\exists!\psi"'] & Y, \end{tikzcd}$$

and we need to show that ψ induces a morphism between C-pairs (Q, D_Q) and (Y, D_Y). As before, we work with Corollary 9.6 and assume that a diagram of the following form is given,

$$\begin{tikzcd} & \widehat{Q} \arrow[r, "\widehat{\psi}"] \arrow[d, "\delta\text{, adapted}"'] & \widehat{Y} \arrow[d, "\varepsilon\text{, adapted}"] \\ X \arrow[r, two heads, "q\text{, quotient morphism}"'] & Q \arrow[r, "\psi"'] & Y. \end{tikzcd}$$

We need to show that $\widehat{\psi}$ admits pull-back of adapted reflexive differentials. As before, we note that this question is local on Q. We may therefore shrink Q, employ Lemma 2.36 ("Strongly adapted covers exist locally") and assume without loss of generality that there exists an adapted cover $\overline{X} \twoheadrightarrow X$. Let $\check{X} = \check{Q}$ be a suitable component of the normalized fibre product $(\overline{X} \times_Q \widehat{Q})^{\text{norm}}$. We obtain an extended diagram as follows,

$$\begin{tikzcd} \check{X} \arrow[r, equal, "\check{q}=\mathrm{Id}"] \arrow[dd, "\gamma\text{, adapted}"'] & \check{Q} \arrow[r, "\check{\psi}"] \arrow[d, two heads, "\delta'\text{, finite}"'] & \check{Y} \arrow[d, equal] \\ & \widehat{Q} \arrow[r, "\widehat{\psi}"] \arrow[d, "\delta\text{, adapted}"'] & \widehat{Y} \arrow[d, "\varepsilon\text{, adapted}"] \\ X \arrow[r, two heads, "q\text{, quotient morphism}"'] & Q \arrow[r, "\psi"'] & Y. \end{tikzcd}$$

Recall Lemma 9.5 ("Test for pull-back of adapted reflexive differentials"). To show that $\widehat{\psi}$ admits pull-back of adapted reflexive differentials, it suffices to show that $\check{\psi}$ admits pull-back of adapted reflexive differentials: there exist sheaf morphisms

$$\eta : \check{\psi}^*\Big(\Omega^{[\bullet]}_{(Y,D_Y,c)}\Big) \to \Omega^{[\bullet]}_{(Q,D_Q,\delta\circ\delta')}$$

that generically agree with the standard pull-back $d_C\breve{\psi}^+$ of adapted reflexive differentials. But we know by assumption that $\breve{\psi}\circ\breve{q}$ admits pull-back of adapted reflexive differentials: There exist sheaf morphisms

$$\begin{aligned}
\breve{\psi}^*\Big(\Omega^{[\bullet]}_{(Y,D_Y,c)}\Big) &= (\breve{\psi}\circ q)^*\Big(\Omega^{[\bullet]}_{(Y,D_Y,c)}\Big) && \breve{q} = \mathrm{Id}\\
&\to \Omega^{[\bullet]}_{(X,D_X,\gamma)} && \breve{\psi}\circ\breve{q}\text{ admits pull-back}\\
&= \Omega^{[\bullet]}_{(Q,D_Q,q\circ\gamma)} && \text{Lemma 12.11}\\
&= \Omega^{[\bullet]}_{(Q,D_Q,\delta\circ\delta')} && \text{Commutativity}
\end{aligned}$$

The claim thus follows. □

13. Pull-back and the Minimal Model Program

13.1. **Pull-back of adapted reflexive tensors.** In Section 8, we defined morphisms of C-pairs are morphisms of varieties where every diagram of form (7.1.1) admits pull-back of adapted reflexive differentials. This section discusses criteria to guarantee that diagrams also admit pull-back of adapted reflexive *tensors*. The main result, formulated in Proposition 13.1 below, relates C-morphisms to notions of minimal model theory, and gives severe restrictions for the existence of resolutions of singularities in the context of C-pairs; we discuss these issues in Section 13.2 right after formulating the main result.

Proposition 13.1 (Criterion for pull-back of adapted reflexive tensors)**.** *Given a morphism $\varphi : (X, D_X) \to (Y, D_Y)$ of C-pairs, let $p \in \mathbb{N}^+$ be any number. Assume that there exists an open covering $Y = \cup Y_i$ and adapted covers $\gamma_i : \widehat{Y}_i \twoheadrightarrow Y_i$ where the sheaves $\Omega^{[p]}_{(Y,D_Y,\gamma_i)}$ of adapted reflexive differentials are locally free. Then, every diagram of form (7.1.1) admits pull-back of adapted reflexive (n, p)-tensors, for every $n \in \mathbb{N}^+$.*

We will prove Proposition 13.1 in Section 13.3 on page 53.

13.2. **Relation to the Minimal Model Program.** We highlight one case where the assumptions of Proposition 13.1 are known to hold.

Proposition 13.2 (Pull-back for morphisms to $\mathbb{Q}$-Gorenstein pairs)**.** *Let $\varphi : (X, D_X) \to (Y, D_Y)$ be a morphism of C-pairs where (Y, D_Y) is locally $\mathbb{Q}$-Gorenstein, where $\dim X = \dim Y$, and where φ is generically finite. If $m \in \mathbb{N}^+$ is any number, then there exists a pull-back map*

$$\varphi^*\Big(\omega_Y^{\otimes m}\otimes\mathscr{O}_Y\big(\lfloor m\cdot D_Y\rfloor\big)\Big)^{\vee\vee}\xrightarrow{\textit{pull-back}}\Big(\omega_X^{\otimes m}\otimes\mathscr{O}_X\big(\lfloor m\cdot D_X\rfloor\big)\Big)^{\vee\vee}\tag{13.2.1}$$

whose restriction to

$$\big(Y_{\mathrm{reg}}\setminus\operatorname{supp}D_Y\big)\cap\varphi^{-1}\big(Y_{\mathrm{reg}}\setminus\operatorname{supp}D_Y\big)$$

agrees with the standard pull-back map of Kähler differentials and their symmetric powers.

Remark 13.3. Proposition 13.2 does not assume that (X, D_X) is locally $\mathbb{Q}$-Gorenstein. If canonical divisors exist, then (13.2.1) can be written in the compact form

$$\varphi^*\mathscr{O}_Y\big(m\cdot K_Y+\lfloor m\cdot D_Y\rfloor\big)\xrightarrow{\text{pull-back}}\mathscr{O}_X\big(m\cdot K_X+\lfloor m\cdot D_X\rfloor\big),$$

which might be more familiar to the algebraic geometer.

Proof of Proposition 13.2. Cover Y by Stein open sets, $Y = \cup_i Y_i$, so that canonical divisors K_{Y_i} exist on each of the Y_i. Recall from Lemma 2.36 that the Stein spaces Y_i admit strongly adapted covers $\gamma_i : \widehat{Y}_i \twoheadrightarrow Y_i$. The pull-back divisors $\gamma_i^*(K_{Y_i}+D_Y)$ are then locally $\mathbb{Q}$-Cartier Weil divisors on the $\widehat{Y}_i$. Shrinking the Y_i if necessary, we may assume without loss of generality that $\gamma_i^*(K_{Y_i} + D_Y)$ are $\mathbb{Q}$-Cartier and that suitable multiples are Cartier and linearly trivial,

$$\mathscr{O}_{\widehat{Y}_i}\big(m_i \cdot \gamma_i^*(K_{Y_i} + D_Y)\big) \cong \mathscr{O}_{\widehat{Y}_i}, \quad \text{for suitable } m_i \in \mathbb{N}^+. \tag{13.4.1}$$

Following [Rei80, Cor. 1.9] or [Rei87, Sect. 3.5–3.7], the isomorphisms (13.4.1) can be used to construct index-one-covers, that is, cyclic covers $\beta_i : \check{Y}_i \twoheadrightarrow \widehat{Y}_i$ where $(\gamma_i \circ \beta_i)^*(K_{Y_i} + D_Y)$ is Cartier. Set $\alpha_i := \gamma_i \circ \beta_i$, $n := \dim X$ and observe that

$$\Omega^{[n]}_{(Y,D_Y,\alpha_i)} \cong \mathscr{O}_{\check{Y}_i}\big(\alpha_i^*(K_{Y_i} + D_Y)\big)$$

is locally free. Apply Proposition 13.1 to the trivial diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}_X\text{, } q\text{-morphism}"'] & Y \arrow[d, "\mathrm{Id}_Y\text{, } q\text{-morphism}"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

and recall from Example 4.6 that domain and range of associated the pull-back map

$$\eta : \widehat{\varphi}^*\Big(\operatorname{Sym}^{[m]}_C \Omega^{[n]}_{(Y,D_Y,\mathrm{Id}_Y)}\Big) \to \operatorname{Sym}^{[m]}_C \Omega^{[d]}_{(X,D_X,\mathrm{Id}_X)}$$

are identified as

$$\begin{aligned} \operatorname{Sym}^{[m]}_C \Omega^{[n]}_{(Y,D_Y,\mathrm{Id}_Y)} &= \big(\omega_Y^{\otimes m} \otimes \mathscr{O}_Y(\lfloor m \cdot D_Y \rfloor)\big)^{\vee\vee} \\ \operatorname{Sym}^{[m]}_C \Omega^{[n]}_{(X,D_X,\mathrm{Id}_X)} &= \big(\omega_X^{\otimes m} \otimes \mathscr{O}_X(\lfloor m \cdot D_X \rfloor)\big)^{\vee\vee} \end{aligned}$$

The proof is thus finished. □

Proposition 13.2 relates the notion of a C-morphism to the notion of discrepancies used in birational geometry, cf. [Rei87, Chapt. I.1] or [KM98, Def. 2.22]. Instead of going into details, we note only one immediate consequence, which refines Proposition 10.6 on page 44.

Corollary 13.5 (C-morphisms and canonical singularities, compare Proposition 10.6)**.** *Let (X, D) be a C-pair that is locally $\mathbb{Q}$-Gorenstein. If a C-resolution of singularities exists, then (X, D) is log canonical. In particular, X has Du Bois singularities.*

Proof. Since the question is local on X, we may assume without loss of generality that (X, D) is $\mathbb{Q}$-Gorenstein and that a canonical divisor exists. Let $\pi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a C-resolution of singularities, with π-exceptional divisor $E \in \operatorname{Div}(\widetilde{X})$. Let $m \in \mathbb{N}^+$ be a number such that $m \cdot D$ is integral and $m \cdot (K_X + D)$ is Cartier. Following Remark 13.3, we obtain a pull-back map

$$\begin{aligned} \pi^*\mathscr{O}_X\big(m \cdot K_X + m \cdot D\big) &\to \mathscr{O}_{\widetilde{X}}\big(m \cdot K_{\widetilde{X}} + m \cdot \widetilde{D}\big) && \text{pull-back} \\ &\subseteq \mathscr{O}_{\widetilde{X}}\big(m \cdot K_{\widetilde{X}} + m \cdot \pi_*^{-1}D + m \cdot E\big), \end{aligned}$$

where $\pi_*^{-1}D$ denotes the strict transform. As in the proof of Proposition 10.6, this means that $\operatorname{discrep}(X, D) \geq -1$, so that (X, D) is log canonical as claimed. □

Remark 13.6 (Converse of Corollary 13.5?). For the time being, we are unsure if a converse of Corollary 13.5 holds and refer the reader to Section 15.4 for questions and a more detailed discussion.

13.3. **Proof of Proposition 13.1.** We prove Proposition 13.1 under the simplifying assumption that there exists one single adapted cover $\gamma_0 : \widehat{Y}_0 \twoheadrightarrow Y$ where $\Omega^{[p]}_{(Y,D_Y,\gamma_0)}$ is locally free. The proof in the general case is conceptually identical, but notationally more involved. We assume that a number $n \in \mathbb{N}^+$ and a diagram of the form (7.1.1) are given,

$$
\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "{a,\ q\text{-morphism}}"'] & \widehat{Y} \arrow[d, "{b,\ q\text{-morphism}}"] \\
X \arrow[r, "\varphi"'] & Y.
\end{tikzcd}
\tag{13.7.1}
$$

Set

$\check{Y}$:= Galois closure of a connected component of the normalized fibre product $\widehat{Y}_0 \times_Y \widehat{Y}$

$\check{X}$:= connected component of the normalized fibre product $\check{Y} \times_{\widehat{Y}} \widehat{X}$

We obtain an extension of Diagram (13.7.1), as follows,

$$
\begin{tikzcd}
\check{X} \arrow[r, "\check{\varphi}"] \arrow[d, two heads, "{\widehat{a},\ \text{Galois cover}}"'] & \check{Y} \arrow[r, equal] \arrow[d, two heads, "{\widehat{b},\ \text{Galois cover}}"] & \check{Y} \arrow[d, "{\widehat{\gamma}_0,\ q\text{-morphism}}"] \\
\widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, "{a,\ q\text{-morphism}}"'] & \widehat{Y} \arrow[d, "{b,\ q\text{-morphism}}"] & \widehat{Y}_0 \arrow[d, two heads, "{\gamma_0,\ \text{adapted cover}}"] \\
X \arrow[r, "\varphi"'] & Y \arrow[r, equal] & Y.
\end{tikzcd}
$$

The assumption that φ is a morphism of C-pairs equips us with a pull-back morphism of adapted reflexive differentials,

$$\eta : \widehat{\varphi}^* \Omega^{[p]}_{(Y,D_Y,b)} \to \Omega^{[p]}_{(X,D_X,a)}.$$

Using the assumption that $\Omega^{[p]}_{(Y,D_Y,\gamma_0)}$ is locally free, we find that

$$\Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} = \Omega^{[p]}_{(Y,D_Y,\gamma_0\circ\widehat{\gamma}_0)} = \widehat{\gamma}_0^{[*]} \Omega^{[p]}_{(Y,D_Y,\gamma_0)} = \widehat{\gamma}_0^{*} \Omega^{[p]}_{(Y,D_Y,\gamma_0)}$$

is likewise locally free. The morphism η therefore induces a pull-back morphism for the sheaves of reflexive adapted (n,p)-tensors on $\check{Y}$, for every $n \in \mathbb{N}$:

$$
\begin{aligned}
\check{\varphi}^* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} &= \check{\varphi}^* \operatorname{Sym}^{[n]} \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} && b\circ\widehat{b} \text{ adapted, Obs. 4.12} \\
&= \check{\varphi}^* \operatorname{Sym}^{n} \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} && \text{locally free} \\
&= \operatorname{Sym}^{n} \check{\varphi}^* \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} && \text{natural} \\
&\to \operatorname{Sym}^{n} \Omega^{[p]}_{(X,D_X,a\circ\widehat{a})} && \operatorname{Sym}^n \eta \\
&\to \operatorname{Sym}^{[n]} \Omega^{[p]}_{(X,D_X,a\circ\widehat{a})} && \text{natural} \\
&\to \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D_X,a\circ\widehat{a})} && \text{Observation 4.8.}
\end{aligned}
\tag{13.7.2}
$$

This in turn yields a morphism between push-forward sheaves,

$$
\begin{aligned}
\widehat{\varphi}^* \widehat{b}_* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} &\to \widehat{a}_* \check{\varphi}^* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})} && \text{natural} \\
&\to \widehat{a}_* \operatorname{Sym}^{[N]}_C \Omega^{[p]}_{(X,D_X,a\circ\widehat{a})}. && \widehat{a}_*(13.7.2)
\end{aligned}
\tag{13.7.3}
$$

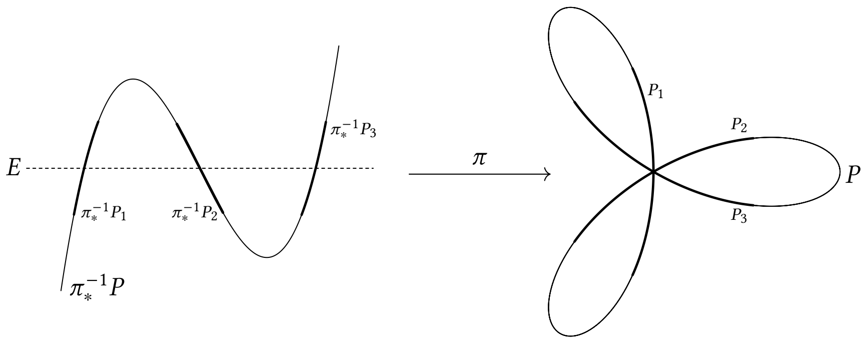

FIGURE 14.1. Blow-up of the *Paquerette de Mélibée*

Recall from Observation 4.19 that all morphisms in (13.7.2) are morphisms of Galois-linearized sheaves. The resulting map (13.7.3) will therefore map Galois-invariant sections to Galois-invariant sections. We obtain the desired pull-back map as follows,

$$
\begin{aligned}
\widehat{\varphi}^* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b)} &= \widehat{\varphi}^* \Bigl(\widehat{b}_* \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(Y,D_Y,b\circ\widehat{b})}\Bigr)^{\operatorname{Galois}(\widehat{b})} && \text{Lemma 4.20} \\
&\to \Bigl(\widehat{a}_* \operatorname{Sym}^{[n]} \Omega^{[p]}_{(X,D_X,a\circ\widehat{a})}\Bigr)^{\operatorname{Galois}(\widehat{a})} && (13.7.3)^{\operatorname{Galois}} \\
&= \operatorname{Sym}^{[n]}_C \Omega^{[p]}_{(X,D_X,a)} && \text{Lemma 4.20.}
\end{aligned}
\tag{13.7.4}
$$

We leave it to the reader to check that the restriction of (13.7.4) to $\widehat{X}^+$ agrees with the pull-back morphism $d_C\widehat{\varphi}^+$, so that Diagram 13.7.1 really does admit pull-back of adapted reflexive (n,p)-tensors, in the sense of Definition 7.6. □

14. Relation to the literature

Morphisms between C-pairs have already been discussed in the literature, but typically only in special settings and for C-pairs that satisfy additional assumptions. While all these notions overlap with Definition 8.1, there is often no implication, unless we are in the simplest setting of C-pairs with snc boundary. For completeness and future reference, this section compares Definition 8.1 with three of the more prominent definitions used in the literature.

14.1. **Orbifold morphisms in the sense of Campana.** Campana uses the following definition in several papers.

Definition 14.1 (Orbifold morphism in the sense of Campana, [Cam11, Déf. 2.3])**.** *Let (X,D_X) and (Y,D_Y) be two C-pairs, where Y is $\mathbb{Q}$-factorial. A morphism $\varphi : X \to Y$ is called* orbifold morphism *if* $\operatorname{img}\varphi \not\subset \operatorname{supp} D_Y$ *and if every pair of prime Weil divisors, $\Delta_X \in \operatorname{Div}(X)$ and $\Delta_Y \in \operatorname{Div}(Y)$ with* $\operatorname{img}\varphi \not\subset \operatorname{supp}\Delta_Y$ *and $\Delta_X \subset \varphi^{-1}(\Delta_Y)$ satisfies the inequality*

$$
\bigl(\operatorname{mult}_{\Delta_X} \varphi^*\Delta_Y\bigr) \cdot \bigl(\operatorname{mult}_{C,\Delta_X} D_X\bigr) \ge \operatorname{mult}_{C,\Delta_Y} D_Y. \tag{14.1.1}
$$

Remark 14.2. The $\mathbb{Q}$-factoriality assumption in Definition 14.1 guarantees that a meaningful pull-back $\varphi^*\Delta_Y \in \mathbb{Q}\operatorname{Div}(X)$ exists.

Definition 14.1 has the drawback that it is not local in the analytic topology. The following examples illustrate some problems.

Example 14.3 (Restriction of orbifold morphism to open set is no orbifold morphism). Let $\pi : X \to Y$ be the blow-up of the affine plane $Y = \mathbb{C}^2$ at the origin, with exceptional divisor $E \subseteq X$. Consider the C-divisors

$$D_Y := \frac{2}{3} \cdot P \in \operatorname{Div}(Y) \quad \text{and} \quad D_X := \frac{2}{3} \cdot \pi_*^{-1} P,$$

where

$$P := \{(x, y) \;:\; (x^2 + y^2)^2 = x^3 - 3 \cdot xy^2\} \in \operatorname{Div}(Y)$$

is the *Paquerette de Mélibée* shown in Figure 14.1. An elementary computation shows that π is an orbifold morphism.

The situation changes if we consider a sufficiently small neighbourhood U of the origin in Y. There, P decomposes into three components, $P = P_1 + P_2 + P_3$, and an elementary computation shows that (14.1.1) is violated when we choose $\Delta_X = E$ and $\Delta_Y = P_1$. It follows that the restricted morphism is not an orbifold morphism in the sense of Definition 14.1.

Example 14.4 (Orbifold morphisms cannot be glued). Let Y be a normal analytic variety that is locally $\mathbb{Q}$-factorial but not $\mathbb{Q}$-factorial. Then, $\operatorname{Id}_Y : Y \to Y$ is not an orbifold endomorphism of the C-pair $(Y, 0)$. Still, Y can be covered by open sets $U \subseteq Y$ such that $\operatorname{Id}_Y|_U$ is an orbifold morphism.

Remark 14.5 (Existence of divisors is not local). We fear that complications similar to those of Example 14.4 arise in settings where Inequality (14.1.1) is void because there are no global divisors in Y that can be used for Δ_Y. This could happen if Y is compact and of algebraic dimension zero.

The authors of this paper feel that non-locality restricts the usefulness of Definition 14.1 in practise. The following alternative notion avoids these problems.

Definition 14.6 (Local orbifold morphism)**.** *Let (X, D_X) and (Y, D_Y) be two C-pairs, where Y is locally $\mathbb{Q}$-factorial. A morphism $\varphi : X \to Y$ is called* local orbifold morphism *if* $\operatorname{img} \varphi \not\subseteq \operatorname{supp} D_Y$ *and if for every pair of sufficiently small open sets $Y^+ \subseteq Y$ and $X^+ \subseteq \varphi^{-1}(Y^+)$, the restricted morphism $\varphi|_{X^+} : X^+ \to Y^+$ is an orbifold morphism in the sense of Definition 14.1.*

Like the notion of a C-morphism, local orbifold morphisms are local in nature and stable under removing small subsets from the source variety. The following analogue of Observation 8.5 on page 36 is not hard to show[9]; its proof is left to the reader.

Observation 14.7 (Local nature and removing small subsets, compare with Observation 8.5). Given C-pairs (X, D_X) and (Y, D_Y) and a morphism $\varphi : X \to Y$ such that $\operatorname{img} \varphi \not\subseteq \operatorname{supp} D_Y$, the following conditions are equivalent.

- The morphism φ is a local orbifold morphism.
- There exist open coverings $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ of X and Y, respectively, such that every restricted morphism
$$\varphi|_{U_i \cap \varphi^{-1}(V_j)} : U_i \cap \varphi^{-1}(V_j) \to V_j$$
is a local orbifold morphism between the restricted C-pairs
$$\big(U_i \cap \varphi^{-1}(V_j),\, D_X \cap U_i \cap \varphi^{-1}(V_j)\big) \quad \text{and} \quad (V_j, D_Y \cap V_j).$$
- There exists a big open subset $U \subset X$ such that the restriction $\varphi|_U$ is a local orbifold morphism $(U, D_X \cap U) \to (Y, D_Y)$. □

The notions "C-morphism" and "local orbifold morphism" are related but not identical. The following propositions compare the notions. Together, they imply that C-morphisms and local orbifold morphisms agree whenever the target pair is nc.

[9] But watch out: Irreducibility is not a local property in the analytic topology.

Proposition 14.8 (C-morphism to locally $\mathbb{Q}$-factorial target). *Let (X, D_X) and (Y, D_Y) be two C-pairs where Y is locally $\mathbb{Q}$-factorial. Then, every C-morphism $\varphi : X \to Y$ with* $\operatorname{img}\varphi \not\subseteq \operatorname{supp} D_Y$ *is a local orbifold morphism.*

Proposition 14.9 (Local orbifold morphism to nc target). *Let (X, D_X) and (Y, D_Y) be two C-pairs. If the pair (Y, D_Y) is nc, then every local orbifold morphism $\varphi : X \to Y$ is a C-morphism.*

Before proving Propositions 14.8 and 14.9 below, we show by way of an elementary example that local orbifold morphisms need not be C-morphisms in general, even if the target is uniformizable.

Example 14.10 (Local orbifold morphisms need not be C-morphisms). Consider the space $Y = \mathbb{C}^2$ and let $D_Y = \frac{2}{3} \cdot D_1 + \frac{2}{3} \cdot D_2 + \frac{1}{2} \cdot D_3$ be the union of three lines passing through a common point $y \in Y$. Recall from Example 2.29 on page 8 that (Y, D_Y) is uniformizable.

Let $\varphi : X \to Y$ be the blow-up of $y \in Y$ and set $D_X := \varphi_*^{-1} D_Y + \frac{2}{3} \cdot \operatorname{Exc}\pi$. An elementary computation that we leave to the reader shows that φ is an orbifold morphism for the pairs (X, D_X) and (Y, D_Y). On the other hand, observe that

$$K_X = \varphi^* K_Y + \operatorname{Exc}\pi \quad \text{and} \quad K_X + D_X = \varphi^*(K_Y + D_Y) - \frac{1}{6} \cdot \operatorname{Exc}\pi.$$

The shift in sign implies that the canonical pull-back map $(d\varphi)^{\otimes 6} : \varphi^* \omega_Y^{\otimes 6} \to \omega_X^{\otimes 6}$ does *not* extend to a pull-back map

$$\varphi^* \omega_Y^{\otimes 6}(6 \cdot D_Y) \xrightarrow{\text{pull-back}} \omega_X^{\otimes 6}(6 \cdot D_X)$$

Proposition 13.2 therefore implies that φ is *not* a C-morphism.

Proof of Proposition 14.8. We prove Proposition 14.8 only under the simplifying assumption that $\lfloor D_X \rfloor = 0$ and $\lfloor D_Y \rfloor = 0$. The proof of the general case is conceptually identical but requires additional case-by-case handling.

Step 0: Setup and simplification. Let $\varphi : X \to Y$ be a C-morphism between two C-pairs (X, D_X) and (Y, D_Y), where Y is locally $\mathbb{Q}$-factorial. Given a pair of open sets $Y^+ \subseteq Y$ and $X^+ \subseteq \varphi^{-1}(Y^+)$, where Y^+ is $\mathbb{Q}$-factorial, and prime divisors $\Delta_Y \subset Y^+$ and $\Delta_X \subseteq \varphi^{-1}(\Delta_Y) \cap X^+$, we need to verify Inequality (14.1.1) from above,

$$(\operatorname{mult}_{\Delta_X} \varphi^* \Delta_Y) \cdot (\operatorname{mult}_{C, \Delta_X} D_X) \geq \operatorname{mult}_{C, \Delta_Y} D_Y. \tag{14.11.1}$$

To this end, choose a general point $x \in \operatorname{supp}\Delta_X$ and set $y := \varphi(x)$. For brevity of notation, set

$$m_X := \operatorname{mult}_{C, \Delta_X} D_X \quad \text{and} \quad m_Y := \operatorname{mult}_{C, \Delta_Y} D_Y.$$

Replacing X and Y by suitably small neighbourhoods of x and y, we may assume without loss of generality that the following holds in addition.

(14.11.2) Using the assumption that Y is locally $\mathbb{Q}$-factorial, we assume that there exists a number $r \in \mathbb{N}^+$ and a holomorphic function $f \in H^0(Y^+, \mathscr{O}_Y)$ such that $\operatorname{div} f = r \cdot \Delta_Y$.

(14.11.3) Using the choice that x is general in $\operatorname{supp}\Delta_X$, we assume that the divisor $\varphi^{-1}(\Delta_Y)$ has only one component, so that $\Delta_X = \operatorname{supp}\varphi^{-1}(\Delta_Y)$.

Step 1: Construct a cover of Y. To begin the proof in earnest, choose a component

$$\widehat{Y} \subseteq \text{normalization of } \{(y, z) \in Y \times \mathbb{C} \,:\, z^{r \cdot m_Y} = f(y)\}$$

and consider the associated cover $\gamma_Y : \widehat{Y} \twoheadrightarrow Y$. The following properties hold by construction.

(14.11.4) The morphism γ_Y is étale outside Δ_Y and $\operatorname{Branch}(\gamma_Y) = \Delta_Y$.

(14.11.5) All components of the divisor $\gamma_Y^*(\Delta_Y)$ have multiplicity m_Y.

(14.11.6) The ramification divisor of γ_Y equals $\operatorname{Ramification}(\gamma_Y) = \frac{1}{m_Y} \cdot \gamma_Y^*(\Delta_Y)$.

(14.11.7) There exists a function $\widehat{f} \in H^0\big(\widehat{Y},\, \mathscr{O}_{\widehat{Y}}\big)$ such that $\widehat{f}^{r \cdot m_Y} = f \circ \gamma_Y$.

Using Items (14.11.6) and (14.11.7), recall from Example 3.6 on page 13 that the Kähler differential of $d\widehat{f} \in H^0\big(\widehat{Y},\, \Omega^1_{\widehat{Y}}\big)$ induces an adapted reflexive differential on $\widehat{Y}$, say

$$\sigma_Y \in H^0\Big(\widehat{Y},\, \Omega^{[1]}_{(Y,D_Y,\gamma_Y)}\Big) \subseteq H^0\Big(\widehat{Y},\, \Omega^{[1]}_{\widehat{Y}}\Big).$$

Step 2: Construct a cover of X. Secondly, choose a component $\widehat{X}$ in the normalization of $X \times_Y \widehat{Y}$. We obtain a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, two heads, "\gamma_X"'] & \widehat{Y} \arrow[d, two heads, "\gamma_Y"] \\ X \arrow[r, "\varphi"'] & Y. \end{tikzcd} \tag{14.11.8}$$

Item (14.11.4) implies that the morphism γ_X is étale outside $\varphi^{-1}(\Delta_Y) \overset{(14.11.3)}{=} \Delta_X$. Given that Δ_Y is the zero-set of the function f, this can be formulated as follows.

(14.11.9) The ramification divisor of γ_X is contained in the zero-locus of the function $\widehat{f} \circ \widehat{\varphi}$.

Step 3: An adapted differential on $\widehat{X}$. Since $\widehat{\varphi}$ admits pull-back of adapted reflexive differentials, recall from Observation 7.10 ("Compatibility with pull-back of Kähler differentials") that the Kähler differential $d\big(\widehat{f} \circ \widehat{\varphi}\big) \in H^0\big(\widehat{X},\, \Omega^1_{\widehat{X}}\big)$ induces an adapted reflexive differential on $\widehat{X}$, say

$$\sigma_X \in H^0\Big(\widehat{X},\, \Omega^{[1]}_{(X,D_X,\gamma_X)}\Big) \subseteq H^0\Big(\widehat{X},\, \Omega^{[1]}_{\widehat{X}}\Big).$$

This poses conditions on the vanishing orders of the function $\widehat{f} \circ \widehat{\varphi}$ along prime divisors $\Delta_{\widehat{X}} \subseteq \gamma_X^* \Delta_X$. To make this statement precise, choose one $\Delta_{\widehat{X}}$, and recall that Item (14.11.9) together with Example 3.6 on page 13 implies that

$$\operatorname{mult}_{\Delta_{\widehat{X}}} \operatorname{div}\big(\widehat{f} \circ \widehat{\varphi}\big) \geq \frac{\operatorname{mult}_{\Delta_{\widehat{X}}} \gamma_X^* \Delta_X}{\operatorname{mult}_{C,\Delta_X} D_X}. \tag{14.11.10}$$

But the left side of (14.11.10) can be computed

$$\begin{aligned} \operatorname{mult}_{\Delta_{\widehat{X}}} \operatorname{div}\big(\widehat{f} \circ \widehat{\varphi}\big) &= \frac{\operatorname{mult}_{\Delta_{\widehat{X}}} \operatorname{div}\big(f \circ \gamma_Y \circ \widehat{\varphi}\big)}{m_X \cdot r} && \text{by (14.11.7)} \\ &= \frac{\operatorname{mult}_{\Delta_{\widehat{X}}} \operatorname{div}\big(f \circ \varphi \circ \gamma_X\big)}{m_Y \cdot r} && \text{commutativity of (14.11.8)} \\ &= \frac{\operatorname{mult}_{\Delta_{\widehat{X}}} \gamma_X^* \varphi^* \Delta_Y}{m_Y} && \text{by (14.11.2)} \\ &= \frac{\big(\operatorname{mult}_{\Delta_{\widehat{X}}} \gamma_X^* \Delta_X\big) \cdot \big(\operatorname{mult}_{\Delta_X} \varphi^* \Delta_Y\big)}{m_Y} && \text{functoriality} \end{aligned}$$

Inserting this into (14.11.10), we obtain Inequality (14.11.1), as required to end the proof of Proposition 14.8. □

Proof of Proposition 14.9. Again we prove Proposition 14.9 under the simplifying assumption that $\lfloor D_X \rfloor = 0$ and $\lfloor D_Y \rfloor = 0$. The proof of the general case is conceptually identical but notationally more involved.

Using Observation 14.7 to remove a suitable small subset from X and consider a suitable open cover of X and Y, it suffices to prove Proposition 14.9 under the simplifying assumption that there exist local coordinates $x_\bullet \in \mathscr{O}_X(X)$ and $y_\bullet \in \mathscr{O}_Y(Y)$ such that the following holds.

(14.12.1) The varieties X and Y are simply connected subsets of $\mathbb{C}^\bullet$.

(14.12.2) There exist numbers $n_i \in \mathbb{N}$ such that $D_Y = \sum \frac{n_i-1}{n_i} \cdot \{y_i = 0\}$.
(14.12.3) There exists a number $n \in \mathbb{N}$ such that $D_X = \frac{n-1}{n} \cdot \{x_1 = 0\}$.
(14.12.4) We have $\operatorname{supp} \varphi^* D_Y \subset \{x_1 = 0\}$.

Item (14.12.4) allows writing the morphism φ in coordinates as

$$\varphi : (x_1,\, x_2,\, x_3,\, \ldots) \mapsto (\ldots,\, \underbrace{x_1^{a_i} \cdot f_i(x_1,\, x_2,\, \ldots)}_{i\text{.th position}},\, \ldots), \quad \text{where all } f_\bullet \in \mathscr{O}_X^*(X).$$

Assumption (14.12.1) allows choosing roots $g_\bullet := \sqrt[n_\bullet]{f_\bullet} \in \mathscr{O}_X^*(X)$. Setting $N := n \cdot \prod n_\bullet$, we can then find smooth varieties $\widehat{X}$, $\widehat{Y}$ with coordinates $\widehat{x}_\bullet \in \mathscr{O}_{\widehat{X}}(\widehat{X})$ and $\widehat{y}_\bullet \in \mathscr{O}_{\widehat{Y}}(\widehat{Y})$ and a commutative diagram,

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, twoheadrightarrow, "a\text{, adapted cover}"'] & \widehat{Y} \arrow[d, twoheadrightarrow, "b\text{, strongly adapted cover}"] \\ X \arrow[r, "\varphi"'] & Y, \end{tikzcd}$$

where

$$\begin{aligned} a &: (\widehat{x}_1,\, \widehat{x}_2,\, \widehat{x}_3,\, \ldots) \mapsto (\widehat{x}_1^N\, \widehat{x}_2,\, \widehat{x}_3,\, \ldots) \\ b &: (\widehat{y}_1,\, \widehat{y}_2,\, \widehat{y}_3,\, \ldots) \mapsto (\widehat{y}_1^{n_1},\, \widehat{y}_2^{n_2},\, \widehat{y}_3^{n_3},\, \ldots) \\ \widehat{\varphi} &: (\widehat{x}_1,\, \widehat{x}_2,\, \widehat{x}_3,\, \ldots) \mapsto \Big(\ldots,\, \underbrace{\widehat{x}_i^{\frac{N\cdot a_i}{n_i}} \cdot g_i(\widehat{x}_1^N,\, \widehat{x}_2,\, \widehat{x}_3,\, \ldots)}_{i\text{.th position}},\, \ldots\Big), \end{aligned}$$

so that

$$\Omega^{[1]}_{(X,D_X,a)} = \left\langle \widehat{x}_1^{\frac{N}{n}-1} \cdot d\widehat{x}_1,\, d\widehat{x}_2,\, \ldots \right\rangle \subseteq \Omega^1_{\widehat{X}} \quad \text{and} \quad \Omega^{[1]}_{(Y,D_Y,b)} = \Omega^1_{\widehat{Y}}.$$

With this description of the orbifold cotangent bundles, the following statements are clearly equivalent.

$$\begin{aligned} &\widehat{\varphi} \text{ admits pull-back of adapted reflexive 1-differentials} \\ &\Leftrightarrow \forall i : d\Big(\widehat{x}_i^{\frac{N\cdot a_i}{n_i}} \cdot g_i(\widehat{x}_1^N,\, \widehat{x}_2,\, \widehat{x}_3,\, \ldots)\Big) \in \Omega^{[1]}_{(X,D_X,a)}(\widehat{X}) \\ &\Leftrightarrow \forall i : \frac{N \cdot a_i}{n_i} - 1 \geq \frac{N}{n} - 1 \\ &\Leftrightarrow \forall i : a_i \cdot n \geq n_i \\ &\Leftrightarrow \forall i : \big(\operatorname{mult}_{\{x_1=0\}} \varphi^*\{y_i = 0\}\big) \cdot \big(\operatorname{mult}_{C,\{x_1=0\}} D_X\big) \geq \operatorname{mult}_{C,\{y_i=0\}} D_Y \\ &\Leftrightarrow \varphi \text{ is an orbifold morphism} \end{aligned} \tag{14.12.5}$$

To finish the proof of Proposition 14.9, recall that $\Omega^{[1]}_{(Y,D_Y,b)} = \Omega^1_{\widehat{Y}}$ is locally free. Proposition 9.3 therefore applies: to show that φ is a $\mathcal{C}$-morphism, it suffices to show $\widehat{\varphi}$ admits pull-back of adapted reflexive 1-differentials. That, however, follows from the equivalences (14.12.5). □

14.2. **Orbifold morphisms in the sense of Lu.** Morphisms of smooth $\mathcal{C}$-pairs have been considered by Lu in the influential preprint [Lu02]. At first glance, Lu's definition [Lu02, Def. 3.7] differs conceptually from Definition 8.1: Given snc $\mathcal{C}$-pairs (X, D_X), (Y, D_Y) and a morphism $\varphi : X \to Y$, Lu's definition of "orbifold morphism" considers open sets $U \subseteq Y$, commutative diagrams of the form

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\varphi|_{\varphi^{-1}(U)}"] \arrow[dr, "a\text{, holomorphic}"'] & U \arrow[d, dashed, "b\text{, meromorphic}"] \\ & Z \end{tikzcd}$$

and compares the saturation of $a^*\omega_Z$ with the φ-pull back of the saturation of $b^*\omega_Z$. However, later in his paper Lu characterizes "orbifold morphisms" in terms that are close to the ideas pursued here. We reformulate his characterization in the language of the present paper.

Proposition 14.13 (Characterization of orbifold morphisms in the sense of Lu, [Lu02, Prop. 4.4])**.** *Let (X, D_X) and (Y, D_Y) be snc C-pairs where X and Y are projective. If $\varphi : X \to Y$ is any morphism with $\operatorname{supp} \varphi^{-1} D_Y \subseteq \operatorname{supp} D_X$, then there exists a commutative diagram of the following form,*

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\varphi}"] \arrow[d, two heads, "a\text{, adapted cover}"'] & \widehat{Y} \arrow[d, two heads, "b\text{, adapted cover}"] \\ X \arrow[r, "\varphi"'] & Y, \end{tikzcd}$$

where $\widehat{X}$ and $\widehat{Y}$ are smooth. Then, φ is an orbifold morphism if and only if the pull-back morphism

$$d\varphi : \widehat{\varphi}^*\left(\Omega^1_{(Y,D_Y,b)}\right) \to \Omega^1_{\widehat{X}}(\log a^{-1}\lfloor D_X \rfloor)$$

factors through the inclusion $\Omega^1_{(X,D_X,a)} \hookrightarrow \Omega^1_{\widehat{X}}(\log a^{-1}\lfloor D_X \rfloor)$. □

Using that all spaces and pairs in Lu's setting are smooth, the criterion for C-morphisms spelled out in Proposition 9.3 on page 39 shows that orbifold morphisms in the sense of Lu are morphisms of C-pairs indeed.

14.3. ***B*-birational morphisms in the sense of Fujino.** For his proof of the abundance theorem for semilog canonical threefolds, Fujino introduced "B-birational morphisms" of pairs. Given the potential for confusion between the various notions of "morphisms of pairs", we briefly recall the definition even though Fujino's notion is essentially unrelated to the C-morphisms discussed here.

Definition 14.14 (B-birational morphisms in the sense of Fujino, [Fuj00, Def. 1.5])**.** *Let (X, D_X) and (Y, D_Y) be two $\mathbb{Q}$-Gorenstein pairs where X and Y are projective. A birational map $\varphi : X \dashrightarrow Y$ is called "B-birational" if there exists a common resolution of singularities*

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "\alpha\text{, resolution}"'] & Z \arrow[d, "\beta\text{, resolution}"] \\ X \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

such that the following equality of $\mathbb{Q}$-divisors on Z,

$$\alpha^*\left(\alpha_* K_Z + D_X\right) = \beta^*\left(\beta_* K_Z + D_Y\right),$$

holds for one (equivalently: every) canonical divisor $K_Z \in \operatorname{Div}(Z)$.

It is easily seen that B-birational maps are hardly ever morphisms of C-pairs.

Example 14.15 (B-birational morphism, not a C-morphism). Consider the space $Y = \mathbb{P}^2$ and let D_Y be a line passing through a point $y \in Y$. Let $\varphi : X \to Y$ be the blow-up of

$y \in Y$ and set $D_X := \varphi_*^{-1} D_Y$. Choose $Z := X$, $\alpha := \mathrm{Id}_X$, and $\beta := \varphi$. Fixing one canonical divisor $K_Y \in \mathrm{Div}(Y)$, an elementary computation shows that

$$\alpha^*(\alpha_* K_Z + D_X) = K_X + D_X = \varphi^*(K_Y + D_Y) = \beta^*\big(\beta_* K_Z + D_Y\big).$$

It follows that φ is a B-birational morphism. However, φ is not a morphism of C-pairs.

15. Problems and open questions

As pointed out in the introduction, this paper is the first in a series [KR24a, KR24b] that develops the theory of C-pairs with a view towards hyperbolicity questions, entire curves and rational points. Still, we see many other interesting directions of research and feel that C-pairs and their adapted reflexive tensors are far from understood. We close with a few of questions and problems that we cannot answer at present.

15.1. **Adapted reflexive tensors.** We feel that the pull-back results presented in Section 5 might not be optimal. A "bigger picture" is still missing.

Question 15.1 (Pull-back for C-pairs with mild singularities). Consider a setup analogous to Setting 5.2: Let (X, D_X) be a C-pair, let (Y, D_Y) be a nc log pair, and consider a sequence of morphisms

$$Y \xrightarrow{\varphi,\text{ resolution of sings.}} \widehat{X} \xrightarrow{\gamma,\ q\text{-morphism}} X,$$

where $\operatorname{supp} \varphi^* \gamma^* \lfloor D_X \rfloor \subseteq \operatorname{supp} D_Y$. We ask for conditions to guarantee that a natural pull-back morphisms for adapted reflexive differentials,

$$d_C\varphi : \varphi^* \Omega^{[p]}_{(X, D_X, \gamma)} \to \Omega^p_Y(\log D_Y)$$

exists for some or all values of p, with universal properties similar to those discussed in Section 5.5?

(15.1.1) Do pull-back maps exist if (X, D_X) is klt?
(15.1.2) What can we say if (X, D_X) is log canonical?
(15.1.3) In analogy to the results obtained in [KS21], is there a natural class of pairs ("pairs with C-rational singularities") that behave optimally with respect to pull-back?

Remark 15.2 (Partial results). In case where $\widehat{X} = X$ and $\gamma = \mathrm{Id}_X$, the papers [GKKP11, Keb13, KS21] answer Questions (15.1.1) and (15.1.2) in the positive. These results are however insufficient to establish a meaningful theory for C-pairs.

Question 15.3 (Pull-back for forms of small degree). Maintaining the setup of Question 15.1, we expect that adapted reflexive p-forms become easier to pull-back, the smaller the value of p.

(15.3.1) Are there results for particularly small values of p that can be seen as C-analogues of the earlier results [vSS85, Fle88]?
(15.3.2) Are there results of the form "the pull-back behaviour of adapted reflexive p-forms follows the extension behaviour $(p+1)$-forms" that could be seen as analogues of [KS21, Thm. 1.4]?

Remark 15.4 (Partial results). In the special case that $p = 1$, Pedro Núñez has shown in his Ph.D. thesis [Nú23b] that a natural morphism as in (15.1.1) exists. It is however unclear at present, if it satisfies enough universal properties to be useful in real-world applications.

15.2. **Invariants of C-pairs.** As pointed out in Section 6.1, the irregularity is of fundamental importance when we discuss C-analogues of the Albanese in the follow-up paper [KR24a]. Still, there are aspects that we do not fully understand.

Question 15.5 (Irregularities). Let (X, D) be a C-pair where X is compact Kähler. How the irregularities $q(X, D, \gamma)$ depend on the choice of the cover $\gamma : \widehat{X} \twoheadrightarrow X$? Can we say anything about the relation between $q(X, D, \bullet)$ and the local geometry of the covering spaces? If X has rational singularities, is it possible that $q(X, D, \gamma) = 0$ for all covers $\gamma : \widehat{X} \twoheadrightarrow X$ where $\widehat{X}$ has rational singularities, and becomes large only for covers that are more singular?

Concerning Question 15.5, recall from [KM98, Prop. 5.13] that rational singularities cannot cover non-rational ones!

Question 15.6 (Bogomolov-Sommese vanishing on covers of X). We do not expect that Proposition 6.15 is optimal. Are there better results for pairs with mild (but worse than uniformizable) singularities?

Note that any answer to Question 15.1 will also answer 15.6.

Problem 15.7 (Special pairs, topology and arithemtics)**.** *Establish arithmetic, topological and geometric properties of mildly singular C-pairs that are special.*

15.3. **Morphisms of C-pairs.** We have seen in Section 13.2 that C-resolutions of singularities will typically not exist. Still, we wonder if they do exist for mildly singular pairs.

Question 15.8 (C-resolution of singularities, existence). Is there a C-resolution for pairs with mild singularities better than Proposition 10.10? Is there a converse to Corollary 13.5?

Question 15.9 (C-resolution of singularities, properties). If (X, D) is a C and if a C-resolution $(\widetilde{X}, \widetilde{D}) \to (X, D)$ exists, then what is the precise relation between the coefficients of $\widetilde{D}$ and the discrepancies of (X, D)? Can the coefficients be expressed in terms of the local fundamental group?

15.4. **Birational geometry of C-pairs.** Campana has studied meromorphic maps, meromorphic fibrations and bimeromorphic maps of nc C-pairs extensively. We feel that this part of the theory should be extended to singular pairs, in order to tie it up with minimal model theory.

Problem 15.10 (C-bimeromorphic maps)**.** *Develop a theory of C-meromorphic and C-bimeromorphic maps. For mildly singular pairs, prove that relevant invariants are bimeromorphically invariant. Following Campana, characterize special pairs in terms of bimeromorphic fibrations to C-pairs of general type.*

Problem 15.11 (Special pairs)**.** *In line with Problem 15.10, follow Campana and characterize mildly singular special C-pairs in terms of C-bimeromorphic fibrations to C-pairs of general type.*

Problem 15.12 (Core map)**.** *In line with Problem 15.10, follow Campana and establish a core map for mildly singular special C-pairs. Study its properties.*

15.5. **C-pairs in other settings.** C-pairs have been applied successfully in algebraic and arithmetic settings. Still, we feel that a systematic treatment of C-morphisms is lacking in many contexts.

Problem 15.13 (C-pairs in the algebraic setting)**.** *Develop a viable theory of C-pairs for algebraic varieties over algebraically closed (or perhaps: perfect) fields of arbitrary characteristic.*

Problem 15.14 ($\mathcal{C}$-pairs in the arithmetic setting)**.** *Develop a viable theory of $\mathcal{C}$-pairs for algebraic varieties over global fields, perhaps following ideas of* [KPS22].

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References

[Abr09] Dan Abramovich. Birational geometry for number theorists. In *Arithmetic geometry*, volume 8 of *Clay Math. Proc.*, pages 335–373. Amer. Math. Soc., Providence, RI, 2009. Preprint arXiv:math/0701105. ↑ 7

[BKT13] Yohan Brunebarbe, Bruno Klingler, and Burt Totaro. Symmetric differentials and the fundamental group. *Duke Math. J.*, 162(14):2797–2813, 2013. DOI:10.1215/00127094-2381442. Preprint arXiv:1204.6443. ↑ 30

[BS76] Constantin Bănică and Octavian Stănăşilă. *Algebraic methods in the global theory of complex spaces*. Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976. Translated from the Romanian. ↑ 22

[BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995. DOI:10.1515/9783110871746. ↑ 25

[Cam04] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004. DOI:10.5802/aif.2027. ↑ 2, 10, 33

[Cam11] Frédéric Campana. Orbifoldes géométriques spéciales et classification biméromorphe des variétés kählériennes compactes. *J. Inst. Math. Jussieu*, 10(4):809–934, 2011. DOI:10.1017/S1474748010000101. Preprint arXiv:0705.0737v5. ↑ 7, 10, 31, 32, 33, 54

[Car57] Henri Cartan. Quotient d'un espace analytique par un groupe d'automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz, DOI:10.1515/9781400879915-007. ↑ 7, 47

[CDR20] Frédéric Campana, Lionel Darondeau, and Erwan Rousseau. Orbifold hyperbolicity. *Compos. Math.*, 156(8):1664–1698, 2020. DOI:10.1112/s0010437x20007265. Preprint arXiv:1803.10716. ↑ 18

[CKT21] Benoît Claudon, Stefan Kebekus, and Behrouz Taji. Generic positivity and applications to hyperbolicity of moduli spaces. In *Hyperbolicity Properties of Algebraic Varieties*, volume 56 of *Panoramas & Synthèses*, pages 169–208. Société Mathématique de France, 2021. Preprint arXiv:1610.09832. ↑ 10

[CP15] Frédéric Campana and Mihai Păun. Orbifold generic semi-positivity: an application to families of canonically polarized manifolds. *Ann. Inst. Fourier (Grenoble)*, 65(2):835–861, 2015. DOI:10.5802/aif.2945. Preprint arXiv:1303.3169. ↑ 2

[CP19] Frédéric Campana and Mihai Păun. Foliations with positive slopes and birational stability of orbifold cotangent bundles. *Publ. Math. Inst. Hautes Études Sci.*, 129:1–49, 2019. DOI:10.1007/s10240-019-00105-w. Preprint arXiv:1508.02456. ↑ 7, 10, 12

[Dem12] Jean-Pierre Demailly. Complex Analytic and Differential Geometry, September 2012. OpenContent Book, freely available from the author's web site. ↑ 4

[EV92] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992. DOI:10.1007/978-3-0348-8600-0. ↑ 31, 33

[Fle88] Hubert Flenner. Extendability of differential forms on nonisolated singularities. *Invent. Math.*, 94(2):317–326, 1988. DOI:10.1007/BF01394328. ↑ 60

[Fuj00] Osamu Fujino. Abundance theorem for semi log canonical threefolds. *Duke Math. J.*, 102(3):513–532, 2000. DOI:10.1215/S0012-7094-00-10237-2. ↑ 59

[Fuj22] Osamu Fujino. Minimal model program for projective morphisms between complex analytic spaces. Preprint arXiv:2201.11315, January 2022. ↑ 8

[Fuj23] Osamu Fujino. Vanishing theorems for projective morphisms between complex analytic spaces. Preprint arXiv:2205.14801v6, July 2023. ↑ 8

[GGK19] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups. *Geom. Topol.*, 23(4):2051–2124, 2019. DOI:10.2140/gt.2019.23.2051. Preprint arXiv:1704.01408. ↑ 30

[GKK10] Daniel Greb, Stefan Kebekus, and Sándor J. Kovács. Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties. *Compositio Math.*, 146:193–219, January 2010. DOI:10.1112/S0010437X09004321. A slightly extended version is available as arXiv:0808.3647. ↑ 7, 39

[GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell. Differential forms on log canonical spaces. *Inst. Hautes Études Sci. Publ. Math.*, 114(1):87–169, November 2011. DOI:10.1007/s10240-011-0036-0. An extended version with additional graphics is available as arXiv:1003.2913. ↑ 3, 16, 19, 24, 31, 39, 60

[GR70] Hans Grauert and Oswald Riemenschneider. Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. *Invent. Math.*, 11:263–292, 1970. DOI:10.1007/BF01403182. ↑ 5

[GR84] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. DOI:10.1007/978-3-642-69582-7. ↑ 4, 6

[Gra13] Patrick Graf. *Differential forms on singular spaces*. PhD thesis, Albert-Ludwigs-Universität Freiburg, February 2013. URN:urn:nbn:de:bsz:25-opus-89935. ↑ 31, 32

[Gra15] Patrick Graf. Bogomolov–Sommese vanishing on log canonical pairs. *J. Reine Angew. Math.*, 702:109–142, 2015. DOI:10.1515/crelle-2013-0031. Preprint arXiv:1210.0421. ↑ 31, 32

[Gri10] Julien Grivaux. Chern classes in Deligne cohomology for coherent analytic sheaves. *Math. Ann.*, 347(2):249–284, 2010. DOI:10.1007/s00208-009-0430-9. Preprint arXiv:0712.2207. ↑ 17

[GT22] Henri Guenancia and Behrouz Taji. Orbifold stability and Miyaoka-Yau inequality for minimal pairs. *Geom. Topol.*, 26(4):1435–1482, 2022. DOI:10.2140/gt.2022.26.1435. Preprint arXiv:1611.05981. ↑ 18

[HP19] Andreas Höring and Thomas Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. *Invent. Math.*, 216(2):395–419, 2019. DOI:10.1007/s00222-018-00853-2. Preprint arXiv:1710.06183. ↑ 30

[JK11] Kelly Jabbusch and Stefan Kebekus. Families over special base manifolds and a conjecture of Campana. *Math. Z.*, 269(3-4):847–878, 2011. DOI:10.1007/s00209-010-0758-6. Preprint arXiv:0905.1746. ↑ 7, 30

[Keb13] Stefan Kebekus. Pull-back morphisms for reflexive differential forms. *Adv. Math.*, 245:78–112, 2013. DOI:10.1016/j.aim.2013.06.013. Preprint arXiv:1210.3255. ↑ 3, 25, 29, 60

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. DOI:10.1017/CBO9780511662560. ↑ 8, 44, 52, 61

[KPS22] Stefan Kebekus, Jorge V. Pereira, and Arne Smeets. Failure of the Brauer–Manin principle for a simply connected fourfold over a global function field, via orbifold Mordell. *Duke Math. J.*, 171(17):3515–3591, 2022. DOI:10.1215/00127094-2022-0045. Preprint arXiv:1307.5718. ↑ 4, 10, 62

[KR24a] Stefan Kebekus and Erwan Rousseau. The Albanese of a C-pair. unpublished preprint, 2024. ↑ 3, 29, 33, 38, 60, 61

[KR24b] Stefan Kebekus and Erwan Rousseau. Entire curves in C-pairs with large irregularity. unpublished preprint, 2024. ↑ 4, 60

[KS21] Stefan Kebekus and Christian Schnell. Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities. *J. Amer. Math. Soc.*, 34(2):315––368, April 2021. DOI:10.1090/jams/962. Preprint arXiv:1811.03644. ↑ 3, 19, 29, 31, 38, 39, 44, 45, 60

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2004. DOI:10.1007/978-3-642-18808-4. ↑ 9

[Lu02] Stefan S.Y. Lu. A refined Kodaira dimension and its canonical fibration. Preprint arXiv:math/0211029, November 2002. ↑ 58, 59

[Miy08] Yoichi Miyaoka. The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem. *Publ. Res. Inst. Math. Sci.*, 44(2):403–417, 2008. DOI:10.2977/prims/1210167331. ↑ 2, 10, 13, 14, 18

[Mum61] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Inst. Hautes Études Sci. Publ. Math.*, (9):5–22, 1961. DOI:10.1007/BF02698717. ↑ 8

[NW14] Junjiro Noguchi and Jörg Winkelmann. *Nevanlinna theory in several complex variables and Diophantine approximation*, volume 350 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Tokyo, 2014. DOI:10.1007/978-4-431-54571-2. ↑ 4

[Nú23a] Pedro Núñez. Adapted differentials as a qfh-sheaf. *Math. Z.*, 304(68):1432–1823, 2023. DOI:10.1007/s00209-023-03330-6. Preprint arXiv:2210.07968. ↑ 10

[Nú23b] Pedro Núñez. *Categorical aspects of Campana orbifolds*. PhD thesis, University of Freiburg, July 2023. DOI:10.6094/UNIFR/237664. ↑ 60

[OSS11] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011. Corrected reprint of the 1988 edition, with an appendix by S. I. Gelfand. DOI:10.1007/978-3-0348-0151-5. ↑ 32, 39

[Rei80] Miles Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980. ↑ 52

[Rei83] Miles Reid. Projective morphisms according to Kawamata. Preprint, University of Warwick. Available from the author's home page., 1983. ↑ 25

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987. ↑ 52

[Ros68] Hugo Rossi. Picard variety of an isolated singular point. *Rice Univ. Studies*, 54(4):63–73, 1968. Available on the internet at https://scholarship.rice.edu/handle/1911/62964. ↑ 5

[Sch92] Michael Schneider. Symmetric differential forms as embedding obstructions and vanishing theorems. *J. Algebraic Geom.*, 1(2):175–181, 1992. ↑ 30

[Ser66] Jean-Pierre Serre. Prolongement de faisceaux analytiques cohérents. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):363–374, 1966. DOI:10.5802/aif.234. ↑ 5, 20, 22

[Sta21] The Stacks Project Authors. Stacks project. Available on the internet at stacks.math.columbia.edu, 2021. ↑ 4

[TT86] Domingo Toledo and Yue Lin L. Tong. Green's theory of Chern classes and the Riemann-Roch formula. In *The Lefschetz centennial conference, Part I (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 261–275. Amer. Math. Soc., Providence, RI, 1986. DOI:10.1090/conm/058.1/860421. ↑ 17

[Ulu07] Muhammed Uludağ. Orbifolds and their uniformization. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 373–406. Birkhäuser, Basel, 2007. DOI:10.1007/978-3-7643-8284-1_13. ↑ 8

[vSS85] Duko van Straten and Joseph Steenbrink. Extendability of holomorphic differential forms near isolated hypersurface singularities. *Abh. Math. Sem. Univ. Hamburg*, 55:97–110, 1985. DOI:10.1007/BF02941491. ↑ 60

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