HYPERBOLICITY AND SPECIALNESS OF SYMMETRIC POWERS

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Abstract. Inspired by the computation of the Kodaira dimension of symmetric powers X_m of a complex projective variety X of dimension $n \geq 2$ by Arapura and Archava, we study their analytic and algebraic hyperbolicity properties. First we show that some (or equivalently any) X_m is rationally connected (resp. special) if and only if so is X (except when the core of X is a curve in the case of specialness). Then we construct dense entire curves in (sufficiently high) symmetric powers of K3 surfaces and product of curves. We also give a criterion based on the positivity of jet differentials bundles that implies pseudo-hyperbolicity of symmetric powers. As an application, we obtain the Kobayashi hyperbolicity of symmetric powers of generic projective hypersurfaces of sufficiently high degree. On the algebraic side, we give a criterion implying that subvarieties of codimension $\leq n-2$ of symmetric powers are of general type. This applies in particular to varieties with ample cotangent bundles. Finally, we use a metric approach to study symmetric powers of ball quotients.

Contents

1. Introduction

For any smooth complex projective variety X with $n = \dim X \geq 2$, and an integer $m \geq 1$, let X_m be the its m-th symmetric power, defined as the quotient of the product X^m of m copies of X by the m-th symmetric group $\mathfrak{S}_m$ acting by permutation of the factors. It is shown in [AA03] that under our assumption that $n \geq 2$, the singularities of X_m are canonical; this implies that if $k = \kappa(X)$ is the

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Kodaira dimension of X, then the Kodaira dimension of any smooth model of X_m is equal to mk. In particular, X is of general type, i.e. $k = n$, if and only if X_m and its smooth models are of general type, i.e. $\kappa(X_m) = nm$. Now, the Green-Griffiths-Lang conjecture claims that a given variety is of general type if and only if it satisfies strong hyperbolicity properties with respect to entire curves or rational points:

Conjecture 1.1 (Green-Griffiths [GG80], Lang [Lan87])**.** *Let X be a smooth projective manifold. Then the following are equivalent:*

(1) X is of general type*;*
(2) X is pseudo-hyperbolic *i.e. there exists a proper algebraic subset $Z \subsetneq X$ that contains the images of all* entire curves*, that is, all holomorphic non-constant maps $f : \mathbb{C} \to X$;*
(3) if X is defined over a number field k, then X is pseudo-arithmetically hyperbolic *i.e. there exists a proper algebraic subset $Z \subsetneq X$ such that $X - Z$ contains finitely many K-rational points for any finite extension K/k.*

Note that the three properties appearing in Conjecture 1.1 are birationally invariant among smooth projective manifolds. In view of the main result of [AA03], this conjecture implies that a symmetric power of a variety of general type and of dimension higher than 2, should also be pseudo-hyperbolic. More precisely, the following conjecture should be true.

Conjecture 1.2. *Let X be a complex projective variety with $n = \dim X \geq 2$. Then X is pseudo-hyperbolic if and only if X_m is pseudo-hyperbolic for some, or any, $m \geq 1$.*

Note that it is not necessary to ask for X to be smooth in the previous conjecture, since pseudo-hyperbolicity is an invariant property by resolution of singularities. Remark also that if X_m is pseudo-hyperbolic for some m, so is X^m, and thus X, so the interesting question is to show that X_m is pseudo-hyperbolic if X is.

The second author has proposed generalizations of the Green-Griffiths-Lang conjectures to any X based on the *specialness* property and the associated core fibration. Special varieties are opposite to varieties of general type in the following sense: they do not admit any fibration with (orbifold) base of general type, or equivalently their core is of dimension 0 (see Section 3, and [Cam04] for details on special varieties and the core map). Conjecturally, special varieties should satisfy exact opposites of the last two points of Conjecture 1.1:

Conjecture 1.3 ([Cam04])**.** *Let X be a complex projective manifold. The following are equivalent:*

(1) X is special*;*
(2) X admits Zariski dense entire curves;
(3) if X is defined over a number field, X admits a potentially dense set of rational points.

Our first goal will be to study the counterpart of Conjecture 1.2 for the specialness property. Accordingly, we were able to derive the following result concerning the specialness of symmetric powers from a study of the canonical fibrations of these varieties (see Section 4):

Theorem 1. *Let X be a complex projective manifold of dimension $n \geq 2$. If X is special then so is X_m for any $m > 0$. Conversely, if X_m is special for some $m > 0$ then either X is special, or the core of X is an orbifold curve of general type of genus at most m.*

Theorem 1 follows from Theorem 12, Theorem 13, and Corollary 4.6 proved in Section 4 below. We give there, more generally, a description of the core map of X_m in terms of the core map of X.

Basic examples of special manifolds are those which are either rationally connected, or with zero Kodaira dimension, generalizing rational and elliptic curves respectively. The Kodaira dimension vanishes for X if and only if the same holds for some (or any) X_m when $\dim X \geq 2$. Similarly:

Theorem 2. *The complex projective manifold X is rationally connected if so is some (or all) X_m.*

Theorem 2 will be obtained as a byproduct of our more precise Corollary 4.2 in Section 4. In view of Conjecture 1.3, this result implies that one should expect corresponding anti-hyperbolicity properties for their symmetric powers. The arithmetic version has already been studied in [HT00b] where the authors prove potential density of rational points in the g-th symmetric power of generic K3 surfaces of degree g. In this article, we will focus on the analytic part, showing that these symmetric powers contain *dense entire curves*, and are even dominated by $\mathbb{C}^{2g}$ (see Theorem 15).

In the case of products of curves, we can also obtain the following result:

Theorem 3. *Let G and C be projective smooth curves of genus $g(G) \leq 1$ and $g(C) > 1$, and let $S = G \times C$. Then $m \geq g(C)$ if and only if S_m contains dense entire curves.*

Note that $m \geq g(C)$ exactly means that S_m is special; this result will be obtained as our Theorem 14 in Section 5. As recently observed in a manuscript sent to us by A. Levin [Lev], such symmetric powers provide negative answers to Puncturing Problems as formulated by Hassett and Tschinkel in [HT01] in the arithmetic and geometric setting, and which can be stated in the analytic setting as follows.

Problem 1.4. *(Analytic Puncturing Problem) Let X be a projective variety with canonical singularities and let Z be a subvariety of codimension at least 2. Assume that there are Zariski dense entire curves on X. Is there a Zariski dense entire curve on $X \setminus Z$?*

In the situation of Theorem 3, considering the small diagonal $Z := \Delta_m \subset S_m$ one easily sees (in Remark 5.2) that Zariski dense entire curves cannot avoid Z, giving a negative answer to this problem. Notice however that no counter-example to the analytic or arithmetic puncturing problem is known or possibly expected when X is smooth. The intermediate case of terminal singularities seems also to be open.

In the second part of the present paper, we study hyperbolicity properties of symmetric powers. Conjecture 1.2 actually looks quite difficult to solve in full generality; we chose to focus on the following particular case which seems already interesting and nontrivial.

Problem 1.5. *Let X be a complex projective manifold with $\dim X \geq 2$, and let $m \geq 2$. Assume Ω_X is ample. Show that any X_m is pseudo-hyperbolic.*

We provide partial answers to this problem by considering instead of Ω_X the more general jet differentials bundles $E^{GG}_{k,r}\Omega_X$: the sections of the latter correspond to algebraic differential equations, or equivalently to sections of lines bundles on the jet spaces $\pi_k : X^{GG}_k \to X$ (see section 2.3 and [Dem97b] for an introduction to these objects). First, we establish a criterion which ensures strong algebraic degeneracy of entire curves in symmetric powers, meaning that the Zariski closure of the union of entire curves, known as the exceptional set $\mathrm{Exc}(X_m)$, is a proper subvariety.

Theorem 4. *Let X be a complex projective manifold. Let A be a very ample line bundle on X. Let $Z \subsetneq X$, and $k, r, d \in \mathbb{N}^*$. We make the following hypotheses.*

(1) Assume that

$$\mathrm{Bs}\left(H^0(X, E^{GG}_{k,r}\Omega_X \otimes \mathcal{O}(-dA))\right) \subset X^{GG,\mathrm{sing}}_k \cup \pi_k^{-1}(Z).$$

(2) Assume that $\frac{d}{r} > 2m(m-1)$.

Then $\mathrm{Exc}(X_m) \neq X_m$.

In fact, there is a precise description of a proper subvariety containing the exceptional locus (see Theorem 16 for details). Our criterion applies to a lot of situations where the Green-Griffiths jet bundles are known to be sufficiently positive to satisfy the assumption of the base locus in Theorem 4. Thanks to all the recent work around the Kobayashi conjecture [Bro17, Den17, Dem18, RY18, BK19], we know that this applies in particular to generic hypersurfaces of high degree in $\mathbb{P}^{n+1}_{\mathbb{C}}$:

Theorem 5. *Let $n \in \mathbb{N}$, and let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a generic hypersurface of degree $d \geq 1$. Let $m \geq 1$ an integer satisfying:*

$$d \geq (2n-1)^5(2m^2 + 10n - 1).$$

The m-th symmetric power X_m of X is then hyperbolic.

This result will be obtained in Corollary 7.9. Getting back to the general case of complex projective manifold X of dimension n, we establish in Section 8 a criterion ensuring that any subvariety $V \subset X_m$ of $\mathrm{codim} V \leq n-2$ is of general type (see Theorem 20). It applies in particular to varieties with ample cotangent bundle:

Theorem 6. *Let X be a complex projective manifold with $n = \dim X \geq 2$, and let $m \geq 1$ be an integer. Assume Ω_X is ample. Then, any subvariety $V \subseteq X_m$ such that $\mathrm{codim} V \leq n-2$ and $V \not\subset X^{\mathrm{sing}}_m$ is of general type.*

If we believe in the Green-Griffiths-Lang conjecture 1.1, this theorem implies that $\mathrm{codim}\,\mathrm{Exc}(X_m) \geq n-1$ for complex manifolds with Ω_X ample, thus giving in principle a strong restriction on the exceptional locus that can appear in Problem 1.5.

This result already permits to obtain several geometric restrictions on the exceptional locus of non-hyperbolic *algebraic* curves in X_m. We obtain in particular the following result (see Corollary 8.8):

Corollary 1.6. *Let X be a complex projective manifold such that Ω_X is ample. Then, there exist countably many proper algebraic subsets $V_k \subsetneq X_m$ ($k \in \mathbb{N}$) containing the image of any non-hyperbolic algebraic curve, such that $\mathrm{codim}_{X_m}(V_k) \geq n-1$ for all $k \in \mathbb{N}$.*

We also obtain genus estimates for curves lying on X in the spirit of [AA03, Corollary 4]. If $Y \subset X$ is a closed submanifold, we say that a generic point $[y_1, \dots, y_l, x_1, \dots, x_{d-l}] \in Y_l \times X_{d-l}$ lies on an irreducible curve with genus g normalization if there exist $\mathcal{C} \to V$ a family of smooth projective curves of genus g and a morphism $f : \mathcal{C} \longrightarrow X$ which is generically one-to-one on the fibers $\mathcal{C}_t$, such that the image Z of $Y_l \times X_{d-l} \to X_d$ is dominated by the image of $S^d f : S^d\mathcal{C} \to X_d$.

Corollary 1.7. *Assume that Ω_X is ample, and let $Y \subset X$ be a closed submanifold. Let $1 \le l \le d$ be integers. Assume that for a generic point $[y_1, \dots, y_l, x_1, \dots, x_{d-l}] \in Y_l \times X_{d-l}$, there exists a curve of geometric genus g in X such that all x_i and y_j lie in C. Then if*

$$l \cdot \mathrm{codim} Y \le \dim X - 2,$$

we have $g > d$.

This result will be proved in Corollary 8.9. Finally, in Section 9, we give a criterion for hyperbolicity in terms of the existence of a suitable negatively curved metric; this criterion applies in particular to symmetric powers of quotient of bounded symmetric domains. As an application, we obtain a hyperbolicity theorem for symmetric products of ball quotients. Before stating it, recall that given a torsion-free lattice with unipotent parabolic elements $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ $(n \in \mathbb{N})$, Mok has given a general construction of *smooth minimal compactification* $\overline{X}$ of the quotient $X = {}_{\Gamma}\backslash^{\mathbb{B}^n}$ (see [Mok12]). The manifold $\overline{X}$ is obtained from X by adding to it a finite union of abelian varieties, forming a boundary divisor D.

In the statement of the theorem (which will be proved as Corollary 9.8), we make use of the following notation: if $W \subset X$ is a subvariety of a variety X, and if $1 \le i \le m$ are integers, we let $\mathfrak{d}_i(W) = \{[x_1, ..., x_m] \in X_m | x_1, \dots, x_i \in W\} \subset X_m$ (see our notation in Section 2.1).

Theorem 7. *Let $X = {}_{\Gamma}\backslash^{\mathbb{B}^n}$ be a ball quotient by a torsion free lattice with only unipotent parabolic elements, and let $\overline{X} = X \cup D$ be a smooth minimal compactification. Let $m \ge 1$. Then :*

(a) Let $V \subset \overline{X}_m$ be a subvariety with $\mathrm{codim} V \le n - 6$ and $V \not\subset \mathfrak{d}_1(D) \cup (\overline{X}_m)_{\mathrm{sing}}$. Then V is of general type.

(b) Let $p \ge n(m-1) + 6$, and $f : \mathbb{C}^p \to \overline{X}_m$ be a holomorphic map such that $f(\mathbb{C}^p) \not\subset \mathfrak{d}_1(D) \cup (\overline{X}_m)_{\mathrm{sing}}$. Then $\mathrm{Jac}(f)$ is identically degenerate.

The paper is organized as follows. In Section 2 we collect some preliminary definitions and properties of symmetric powers and jet differentials. In Section 3 we recall the basic definitions and constructions related to special varieties. In Section 4 we prove Theorem 1 and Theorem 2. In Section 5 we prove Theorem 3. In Section 6 we state some basic facts on Kobayashi hyperbolicity of symmetric powers. In Section 7 we prove Theorem 4 and Theorem 5. In Section 8 we prove Theorem 6, Corollary 1.6 and Corollary 1.7. Finally, in Section 9 we prove Theorem 7.

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2. Notation and conventions

We introduce here some notation pertaining to symmetric powers of manifolds, that we will use in the entirety of the article.

2.1. Symmetric powers.

Let X be a complex projective manifold.

(1) For any $m \in \mathbb{N}^*$, we will denote by $X_m = {\mathfrak{S}_m}\backslash^{X^m}$ the m-th symmetric power of X. We let $q : X^m \to X_m$ be the natural projection. Elements of X_m will be denoted by $[x_1, x_2, ..., x_m]$ (where $(x_1, ..., x_m) \in X^m$). Also, if $s > 0, m_1, \dots, m_s$ are positive integers such that $\sum_i m_i = m$, and $x_1, \dots, x_s \in X$ are pairwise distinct, we write $[x_1^{m_1}, \dots, x_s^{m_s}] := [x_1, \dots, x_1, \dots, x_s, \dots, x_s]$, where each x_i is repeated m_i times, for $i = 1, \dots, s$.

(2) For any $V \subset X$ and any $1 \leq i \leq m$, we let $\mathfrak{d}_i(V) = \{[x_1, ..., x_m] \in X_m | x_1, ..., x_i \in V\}$.

(3) For any $1 \leq i \leq m$, we let $\mathfrak{D}_i(X_m) = \{[x_1, ..., x_m] \in X_m | x_1 = ... = x_i\}$ be the i-th diagonal locus. Note that $\operatorname{codim} \mathfrak{D}_i(X_m) = n(i-1)$.

(4) For any divisor A on X, we will denote by $A^\sharp = \sum_{i=1}^{m} \mathrm{pr}_i^* A$ the associated $\mathfrak{S}_m$-invariant divisor on X^m. Since $A^\sharp$ admits $\mathfrak{S}_m$-invariant local defining equations, the latter are pull-backs of equations on X_m: this means that there exists an effective Cartier divisor $A_\flat$ on X_m such that $q^* A_\flat = A^\sharp$. Note that since $A_\flat$ is a Cartier divisor on X_m, it induces a well-defined line bundle.

Remark that the construction $X \rightsquigarrow X_m$ is functorial, any holomorphic map $f : X \to Y$ inducing a natural holomorphic map $f_m : X_m \longrightarrow Y_m$.

2.2. The Reid-Tai-Weissauer criterion.

For later reference, we now recall an important criterion for the extension of differential forms on resolutions of quotient singularities.

Let G be a finite group acting on a complex manifold X of dimension n. The criterion can be stated in terms of the following condition:

Condition $(\mathrm{I}_{x,d})$**.** *Let $x \in X$, and let $d \in \mathbb{N}$. Let $g \in G$ having order $r > 1$ and stabilizing x. Then there exists coordinates $(z_1, ..., z_n)$, centered at x such that g acts by*

$$g \cdot (z_1, ..., z_n) = (\zeta^{a_1} z_1, ..., \zeta^{a_n} z_n),$$

where $\zeta = e^{\frac{2i\pi}{r}}$, and $a_1, ..., a_n \in [\![0, r-1]\!]$. We say that the condition $(I_{x,d})$ is satisfied, if for any such $g \in G - \{1\}$ stabilizing x, the following holds for any choice of d distinct elements $i_1, ..., i_d$ in $[\![1, n]\!]$:

$$a_{i_1} + ... + a_{i_d} \geq r.$$

Note that it is always possible to find coordinates $z_1, \dots, z_n$ as above by the classical lemma of H. Cartan [Car54]; whether the criterion holds or not is independent on such a choice of coordinates.

It is useful to state a weaker condition under which the differentials will extend meromorphically to a resolution of singularities. Resume the same notation as before, and let $\alpha > 0$.

Condition $(\mathrm{I}'_{x,d,\alpha})$**.** *We say that the condition* $(\mathrm{I}'_{x,d,\alpha})$ *is satisfied, if the same statement as in Condition* $(\mathrm{I}_{x,d})$ *holds, with the inequality replaced by*

$$a_{i_1} + ... + a_{i_d} \geq r(1-\alpha).$$

Proposition 2.1 ([Wei86, Lemma 4. p. 213])**.** *Let* $d \in \mathbb{N}$*. Assume that the condition* $(\mathrm{I}_{x,d})$ *(resp.* $(\mathrm{I}'_{x,d,\alpha})$*) holds for any point* $x \in X$*. Let* $Y = {}_G\backslash^{X}$*, and let* $\widetilde{Y}$ *be a smooth resolution of singularities of* Y*. Let* Y° *be the smooth locus of* Y*.*

Then, for any $p \geq d$*, and for any* $q \in \mathbb{N}$*, the sections of* $(\bigwedge^p \Omega_{Y^\circ})^{\otimes q}$ *extend to the whole* $\widetilde{Y}$ *(resp. extends as meromorphic section of* $(\bigwedge^p \Omega_{\widetilde{Y}})^{\otimes q}$ *with a pole of order at most* $\lfloor q\alpha \rfloor$*).*

Remark 2.2. 1. The fact that q is arbitrary in the criterion above is crucial. Note that if $q = 1$, then for any $p \geq 1$, any section of $\bigwedge^p \Omega_{Y^\circ}$ extends to $\widetilde{Y}$, e.g. by [Fre71] or [GKKP10]. The proof of [Fre71] consists essentially in remarking that $(\mathrm{I'}_{x,d,\alpha})$ always holds for some $\alpha < 1$, so $\lfloor q\alpha \rfloor = 0$ in this case.

2. Proposition 2.1 is a generalization of well-known criterion proved independently by Tai [Tai82] and Reid [Rei79] (which is simply the case $p = \dim X$). The proof given in [Wei86] is stated in the case where $X = \mathbb{H}_g$ is the Siegel upper half-space acted upon by $G = \mathrm{Sp}(2g, \mathbb{Z})$, and where G is a cyclic group ; by an argument of Tai [Tai82, Proposition 3.1], the cyclic case suffices to deal with the general situation, and Weissauer's computations can be adapted immediately to the general case formulated above. For more details in English, the reader can see e.g. [Cad18, Section 4].

2.3. **Jet differentials.** We will now recall some basic facts around the notion of jet differentials. For more details, the reader can refer to [Dem12, §7].

Let X be a complex manifold, and $k, m \in \mathbb{N}$ be integers. We will denote the unit disk by Δ. The *Green-Griffiths vector bundle of jet differentials of order* k *and degree* m, is the vector bundle $E^{GG}_{k,m}\Omega_X \to X$, whose sections over a chart $U \subset X$ identify with differential equations acting on holomorphic maps $f : \Delta \to U$, with adequate order and degree. Writing $f = (f_1, ..., f_n)$ in local coordinates, $P(f)$ can be written as a holomorphic polynomial $P_0(f; f', \dots, f^{(k)})$ in the first k derivatives of the f_i, being of degree m with respect to reparametrization, i.e. $P(g)(t) = \lambda^m P(f)(\lambda t)$ if $g(t) = f(\lambda t)$.

For any order $k \geq 1$, we can form the Green-Griffiths jet differential algebra $E^{GG}_{k,\bullet}\Omega_X = \bigoplus_{m\geq 0} E_{k,m}\Omega_X$, and define the *k-th jet space* $X^{GG}_k = \mathbf{Proj}_X(E^{GG}_{k,\bullet}\Omega_X)$. We check that the elements of X^{GG}_k are naturally identified with *classes of k-jets*, i.e. k-th order Taylor expansions of holomorphic maps $f : (\Delta, 0) \to X$, up to linear reparametrization. Each jet space is endowed with a projection map $\pi_k : X^{GG}_k \to X$ and tautological sheaves $\mathcal{O}_{X^{GG}_k}(m)$ $(m \geq 0)$, such that

$$(\pi_k)_* \mathcal{O}_{X^{GG}_k}(m) = E^{GG}_{k,m}\Omega_X$$

for any $m \geq 1$.

If C is a complex curve, any map $f : C \to X$ admit well-defined lifts $f_{[k]} : C \to X_k^{GG}$ obtained by taking the k-th Taylor expansion at each point of C. The main interest of jet differential equations in the study of complex hyperbolicity comes from the following fundamental vanishing theorem, which permits to give strong restrictions on the geometry of entire curves.

Theorem 8 ([SY96, Dem97a]). *Let X be a complex projective manifold, and let A be an ample line bundle on X. Let $k, m \geq 1$, and let $P \in H^0(X, E_{k,m}^{GG}\Omega \otimes \mathcal{O}(-A))$. Let $f : \mathbb{C} \longrightarrow X$. Then f is a solution of the holomorphic differential equation P, i.e. $P(f; f', ..., f^{(k)}) = 0$.*

In other words, for any entire curve $f : \mathbb{C} \to X$, we have $f_{[k]}(\mathbb{C}) \subset \mathbb{B}_+(\mathcal{O}_{X_k^{GG}}(1))$, where $\mathbb{B}_+$ denotes the augmented base locus.

The previous theorem has strong implications in cases where global jet differential equations are numerous. In these notes, we will be able to produce such differential equations using a basic variant of the *orbifold jet differentials* which were introduced by the second and third authors in a joint work with L. Darondeau [CDR18]. We will explain briefly how these objects can be defined in our context at the beginning of Section 7.1.

Part 1. Specialness of symmetric powers

3. Special varieties

We collect here basic definitions and constructions related to special varieties, while referring to [Cam04] for more details.

3.1. Special Manifolds via Bogomolov sheaves.

Let X be a connected complex nonsingular projective manifold of complex dimension n. For a rank-one coherent subsheaf $\mathcal{L} \subset \Omega_X^p$, denote by $H^0(X, \mathcal{L}^m)$ the space of sections of $\mathrm{Sym}^m(\Omega_X^p)$ which take values in $\mathcal{L}^m$ at the generic point of X (where as usual $\mathcal{L}^m := \mathcal{L}^{\otimes m}$).

The *Iitaka dimension* of $\mathcal{L}$ is $\kappa(X, \mathcal{L}) := \max_{m>0}\{\dim(\Phi_{\mathcal{L}^m}(X))\}$, i.e. the maximum dimension of the image of rational maps $\Phi_{\mathcal{L}^m} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}^m))$ defined at the generic point of X, where by convention $\dim(\Phi_{\mathcal{L}^m}(X)) := -\infty$ if there are no global sections. Thus $\kappa(X, \mathcal{L}) \in \{-\infty, 0, 1, \ldots, \dim(X)\}$. In this setting, a theorem of Bogomolov in [Bog79] shows that, if $\mathcal{L} \subset \Omega_X^p$, then $\kappa(X, \mathcal{L}) \leq p$.

Definition 3.1. *Let $X, p > 0$ as above. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a* Bogomolov sheaf *if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.*

Definition 3.2. *(*[Cam04, Definition 2.1]*) A nonsingular complex projective*[1] *variety X is said to be special (or of special type) if there is no Bogomolov sheaf on X. A projective variety is said to be special if some (or any) of its resolutions are special.*

Bogomolov sheaves on X occur if $f : X \to Y$ is a fibration on Y, of general type and dimension $p > 0$, indeed:

[1]Bogomolov theorem works in the compact Kähler setting as well, and so do the notions of special variety and core map.

Remark 3.3. If $f\colon X\to Y$ is a fibration (by which we mean a surjective morphism with connected fibers) and Y is a variety[2] of general type of dimension $p>0$, then the saturation of $f^*(K_Y)$ in Ω^p_X is a Bogomolov sheaf of X.

By the previous remark if there is a fibration $X\to Y$ with Y of general type then X is nonspecial. In particular, if X is of general type of positive dimension, X is not of special type. However, Bogomolov sheaves occur, more generally, when $f:X\to Y$ fibres over Y, even if Y is not of general type, provided f has enough multiple fibres.

3.2. **Special Manifolds via orbifold bases.** Special varieties are alternatively characterized using the notion of *orbifolds*. We briefly recall the construction.

Let Z be a normal connected compact complex variety. An *orbifold divisor* Δ is a linear combination $\Delta := \sum_{\{D\subset Z\}} c_\Delta(D)\cdot D$, where D ranges over all prime divisors of Z, the *orbifold coefficients* are rational numbers $c_\Delta(D) := (1-\frac{1}{m_\Delta(D)})\in[0,1]\cap\mathbb{Q}$ such that all but finitely many are zero. Equivalently,

$$\Delta=\sum_{\{D\subset Z\}}\left(1-\frac{1}{m_\Delta(D)}\right)\cdot D=\sum_{j\in J}\left(1-\frac{1}{m_j}\right)\cdot D_j,$$

where only finitely *orbifold multiplicities* $m_j := m_\Delta(D_j)\in\mathbb{Q}_{\geq 1}\cup\{+\infty\}$ are larger than 1.

An orbifold pair is a pair (Z,Δ) where Δ is an orbifold divisor; they interpolate between the compact case where $\Delta=\varnothing$ and the pair $(Z,\varnothing)=Z$ has no orbifold structure, and the *open*, or *purely-logarithmic case* where $c_j=1$ for all j, and we identify (Z,Δ) with $Z\setminus \mathrm{Supp}(\Delta)$.

When Z is smooth and the support $\mathrm{Supp}(\Delta):=\cup D_j$ of Δ has normal crossings singularities, we say that (Z,Δ) is *smooth*. When all multiplicities m_j are integral or $+\infty$, we say that the orbifold pair (Z,Δ) is *integral*, and when every m_j is finite it may be thought of as a virtual ramified cover of Z ramifying at order m_j over each of the D_j's.

Consider a fibration $f\colon X\to Z$ between normal connected complex projective varieties. In general, the geometric invariants (such as $\pi_1(X),\kappa(X),\dots$) of X do not coincide with the 'sum' of those of the base (Z) and of the generic fiber (X_η) of f. Replacing Z by the 'orbifold base' (Z,Δ_f) of f, which encodes the multiple fibers of f, leads in some favorable important cases to such an additivity (on suitable birational models at least).

Definition 3.4 (Orbifold base of a fibration)**.** *Let $f:X\to Z$ be a fibration, and let Δ be an orbifold divisor on X. We then write $f\colon(X,\Delta)\to Z$ to indicate that Δ is taken into account. We shall define the* orbifold base *(Z,Δ_f) of (f,Δ) as follows: to each irreducible Weil divisor $D\subset Z$ we assign the multiplicity $m_{(f,\Delta)}(D):=\inf_k\{t_k\cdot m_\Delta(F_k)\}$, where $f^*(D)=\sum_k t_k.F_k+R$, R is an f-exceptional divisor of X with $f(R)\subsetneq D$, and F_k are the irreducible divisors of X which map surjectively to D via f, with fibre of multiplicity t_k over the generic point of D.*

Remark 3.5. Note that the integers t_k are well-defined, even if X and Z are only assumed to be normal.

[2] Y normal is sufficient, by considering $\mathcal{L}:=f^*(i_*(K_{Y^0}))$, where $i:Y^0\to Y$ is the injection of the regular locus of Y.

Let (Z, Δ) be an orbifold pair. Assume that $K_Z + \Delta$ is $\mathbb{Q}$-Cartier (this is the case if (Z, Δ) is smooth, for example): we will call it the *canonical bundle* of (Z, Δ). Similarly we will denote by the *canonical dimension* of (Z, Δ) the Kodaira dimension of $K_Z + \Delta$ i.e. $\kappa(Z, K_Z + \Delta) := \kappa(Z, \mathcal{O}_Z(k.(K_Z + \Delta)))$, for $k > 0$ any integer such that $k.(K_Z + \Delta)$ is Cartier. Finally, we say that the orbifold (Z, Δ) is of *general type* if $\kappa(Z, \Delta) = \dim(Z)$.

Definition 3.6. *A fibration $f\colon X \to Z$ with X, Z projective, X smooth and Z normal, is said to be of general type if (Z, Δ_f) of general type.*

If $f : X \dashrightarrow Z$, $\dim(Z) = p > 0$, is only a rational fibration, we may replace X, Z, f by birational models and assume that (Z, Δ_f) is smooth. The saturated rank-one sheaf $\mathcal{L} \subset \Omega^p_X$ which coincides with $f^*(K_Y)$ over the regular locus of Y has then a well-defined $\kappa(X, \mathcal{L})$ as said in the beginning of the present subsection, easily seen to be independent of the birational models chosen, and can be seen to be equal to $\kappa(Z, K_Z + \Delta_f)$ on any suitably chosen 'neat' birational model of f.

The non-existence of fibrations of general type in the above sense turns out to be equivalent to the specialness condition of Definition 3.2.

Theorem 9 (see [Cam04, Theorem 2.27])**.** *A complex projective manifold X is special if and only if it has no rational fibrations $f : X \dashrightarrow Z$ of general type.*

Let us now recall the existence of the *core* map (see [Cam04, Section 3] for details). Given a smooth projective variety X there is a functorial fibration $c_X : X \to C(X)$, called the *core* of X such that the fibers of c_X are special varieties and the base $C(X)$ is either a point (if and only if X is special) or an orbifold $(C(X), \Delta_{c_X})$ of general type. This 'core map' dominates birationally any fibration $f : X \dashrightarrow Z$ with general type orbifold base, and its fibres are also the largest special subvarieties of X going through the general point of X.

As mentioned in the introduction, the second author has proposed in [Cam04] the following generalizations of Lang's conjectures.

Conjecture 3.7.
(1) Let X be a complex projective variety. Then, X is special if and only if there exists an entire curve $\mathbb{C} \to X$ with Zariski dense image.
(2) Let X be a projective variety defined over a number field. Then, the set of rational points on X is potentially dense if and only if X is special.

Finally, let us remark that previous conjectures (see [HT00a, Conjecture 1.2]) proposed to characterize potential density with the weaker notion of *weak specialness*.

Definition 3.8. *A projective variety X is said to be weakly special if there are no finite étale covers $u\colon X' \to X$ admitting a dominant rational map $f'\colon X' \to Z'$ to a positive dimensional variety Z' of general type.*

It has been shown in [CP07] and [RTJ20] that one cannot replace "special" by "weakly-special" in Conjecture 3.7 in the analytic and function fields settings.

4. Canonical fibrations

We will now study conditions under which various canonical fibrations are preserved by the symmetric product. In the rest of the text, a *fibration* will be a surjective morphism with connected fibres. Then, if $f : X \to B$ is a fibration, so is $f_m : X_m \to B_m$. We denote with $Y_m \to X_m$ any desingularisation of X_m.

We shall consider the following (bimeromorphically well-defined) fibrations for X smooth compact of dimension n:

(1) The Moishezon-Iitaka fibration $J : X \to B$

Assuming X to be smooth compact Kähler:

(2) The 'rational quotient'[3] $r : X \to B$.
(3) The 'core map' $c : X \to B$.

Recall that [AA03] shows that if X is smooth, and if $\dim X \geq 2$, the singularities of X_m are canonical, and consequently, that $\kappa(Y_m) = \kappa(X_m) = m.\kappa(X)$.

The goal is to extend (and exploit) [AA03] in order to show the following:

Theorem 10. *Let X be smooth projective*[4]*, and let $f : X \to B$ be any one of the three canonical fibrations $f = J, r, c$ respectively. Assume $\dim B \geq 2$, then for each of these 3 fibrations, the corresponding fibration of Y_m is nothing but the m-th symmetric product $f_m : Y_m \to B_m$. Explicitly: the Moishezon-Iitaka fibration of Y_m is $J_m : Y_m \to B_m$, the rational quotient map of Y_m is $r_m : Y_m \to B_m$, and the core map of Y_m is $c_m : Y_m \to B_m$. (When B is a curve, a simple description can be given, too. See Theorems 11 and 12 below, as well as Remark 4.1).*

Remark 4.1. The conclusion is obviously false when $\dim X = 1$ and $g(X) \geq 2$, since $q_m : X^m \to X_m$ then ramifies in codimension $n = 1$. One recovers a uniform statement by equipping X_m with its natural orbifold structure, obtained by assigning to each component $D_{j,k}$ in X_m of the diagonal locus $\mathfrak{D}_2(X_m)$ its natural multiplicity 2. The orbifold divisor $D_m := \sum_{j<k}(1 - \frac{1}{2}).D_{j,k}$ on X_m has then the property that $q_m^*(K_{X_m} + D_m) = K_{X^m}$. In particular, $\kappa(X_m, K_{X_m} + D_m) = m.\kappa(X)$. The divisor D_m will appear again when we consider the core map below. Notice however that, as soon as $m \geq 3$, the orbifold divisor D_m is not of normal crossings (for $m = 3$ for example, it is locally analytically a product of of disk by a plane cusp.)

Before starting the study of J_m, c_m, r_m, let us make some simple observations on $f_m : X_m \to B_m$ if $f : X \to B$ is a fibration (with connected fibres) between two connected compact complex manifolds, with $\dim(B) \geq 1$:

1. The generic fibre of f_m over a point $[b_1, \ldots, b_m] \in B_m$ is isomorphic to the (unordered) product $X_{b_1} \times \ldots X_{b_m}$ if the b_i are pairwise distinct. In particular, if the generic fibre of f is rationally connected, or special, so are the generic fibres of f_m. The rational quotient map and the core map of Y_m thus factorise through r_m and c_m respectively.
2. If the schematic fibres X_{b_i} are reduced, so is the fibre over $[b_1, \ldots, b_m]$, whatever the b_i.
3. If f has a local section over a neighborhood of each of the $b_i's$, f_m has (an obvious) local section over a neighborhood of $[b_1, \ldots, b_m]$.

For $f = J$, the proof is an immediate consequence of [AA03]. Indeed: the general fibre of f_m is a product of fibres of J, hence has $\kappa = 0$. On the other hand, $\kappa(X_m) = m.\kappa(X) = \dim(B_m)$. The conclusion follows.

We shall now prove the statement for the two remaining fibrations r, c.

[3] Also termed MRC fibration.

[4] For J (resp. r, c), the statement is still valid for X compact (resp. for X compact Kähler). For J, this is simply due to the fact that [AA03] is purely local in the analytic topology. For r, c this will be explained briefly in the next footnotes.

4.1. **The 'rational quotient'.**

Theorem 11. *Let $r : X \to B$ be the rational quotient map of X, smooth complex projective*[5] *Then $r_m : X_m \to B_m$ is the rational quotient map of X_m if $\dim(B) \neq 1$. If B is a curve of genus $g > 0$, and $R_m : X_m \to R(m)$ is the rational quotient map, there are two cases: either $m < g$, then $R_m = r_m, R(m) = B_m$, or $R_m = jac_B^m \circ r_m : X_m \to \mathrm{Jac}(B)$, where $jac_B^m : B_m \to \mathrm{Jac}(B)$ is the natural Jacobian map.*

Proof. We assume X to be complex projective. Recall that r is characterised by the fact that its fibres are rationally connected and (a smooth model of) its base is not uniruled (by [GHS03]). Since the generic fibres of r_m are products of fibres of r, hence rationally connected, it is sufficient to show that a smooth model $\mu : B'_m \to B_m$ of B_m is not uniruled if B is **not** a curve of positive genus. Assume B'_m were uniruled, we would then have an irreducible algebraic family of curves C'_t covering B'_m and with $-K_{B'_m}.C'_t > 0$. Since the singularities of B_m are canonical, this implies $K_{B_m}.C_t < 0$, where $C_t := \mu_*(C_t)$, since $K_{B'_m} = \mu^*(K_{B_m}) + E'$, with E' effective, by [AA03]. The conclusion[6] now follows, using [MM86], from the fact that $K_{B^m} = (q_m^B)^*(K_{B_m})$ is pseudo-effective (i.e. has nonnegative intersection with any covering algebraic family of generically irreducible curves[7]), by lifting to B^m the generic curve C_t.

Assume now that B is a curve of genus $g > 0$. Then $jac_B^m : B_m \to \mathrm{Jac}(B)$ has connected fibres generically projective spaces of dimension 0 if $m \leq g$, and positive dimension if $m > g$. Moreover the image of jac_B^m is never uniruled when $m > 0$. This shows the claim, by [GHS03].

We now show how to adapt this argument when X is compact Kähler. The rational quotient map (with maximally rationally connected fibres) still exists in the compact Kähler case, by the compactness of the components of the Chow-Barlet 'scheme'. Assume by contradiction that B'_m is uniruled. Let then $r' : B'_m \to R'$, the MRC fibration of B'_m: its generic fibre is thus smooth, positive-dimensional, and rationally connected. From the last part of the preceding argument in the case when B is projective, we conclude that B_m is covered by an analytic family of curves (images of rational curves contained in the fibres of r') with negative intersection with K_{B_m}, and thus that K_{B^m} is not pseudo-effective, contradicting the fact that K_B is pseudo-effective. □

We can now prove Theorem 2 as a corollary of the previous theorem:

Corollary 4.2. *A smooth projective variety X is rationally connected if and only if so is X_m for some m, and X is uniruled if and only if so is X_m for some m, unless X is a curve of genus $g > 0$, and $m > g$.*

Proof. Indeed: the uniruledness (resp. rational connectedness) of X is characterised by: $\dim(X) > \dim(B)$ (resp. $\dim(B) = 0$), and $\dim(B_m) = m.\dim(B)$. We thus see that any X_m is rationally connected (resp. uniruled) if so is X. Conversely, the preceding Theorem 11 shows that the claim holds true if $\dim(R(m)) = \dim(B_m) = m.\dim(X)$. This is the case unless possibly when $r : X \to B$ fibres over a curve

[5] The proof still works for X compact Kähler, as explained below.

[6] One does not really need [MM86], since it is sufficient to show that $K_{B'_m}$ is pseudo-effective.

[7] By this, we mean an irreducible and compact complex space Γ equipped with two surjective holomorphic maps $p : \Gamma \to S$ and $\Gamma \to X$, with $dim(S) + 1 = dim(\Gamma)$.

B with $g(B) > 0$, and $m > g$. In this case, X_m is uniruled, but not rationally connected. Thus X_m rationally connected for some $m > 0$ implies that X rationally connected. On the other hand, if X is not uniruled, we have $X = B$ is a curve, and X_m is uniruled if and only if $m > g$. Hence the corollary. □

Remark 4.3. If X is unirational, so is obviously X_m, for any $m > 1$. It is true, but less obvious ([Mat68], that if X is rational, then so is X_m, for any $m > 1$ (much more is to be found in [CS07] and [Pop13]). From this, it follows that if X is stably rational, then so is X_m, for $m > 1$ too. This naturally leads to ask about the converses.

Question 1. Assume that X_m is unirational (resp. rational, stably rational) for some $m \geq 2$, is then, yes or no, X unirational (resp. rational, stably rational)? If some $X_m, m > 1$ is rational, is X unirational?

Some specific cases are as follows.

Example 1. 1. If X is a smooth cubic hypersurface of dimension $n \geq 3$, is X_m rational for some large m?

2. If X is the double cover of $\mathbb{P}^3$ ramified over a smooth sextic surface, X is Fano, hence rationally connected, but its unirationality (or not) is an open problem. Is X_m unirational for some large m? The same question arises for X a conic bundle over $\mathbb{P}^2$ with a smooth discriminant of large degree.

3. Can the Brauer group of a smooth model of X_m be estimated from the one of X? Does it vanish for m sufficiently large if X is unirational (resp. rationally connected)? To which extent do the Brauer groups of X_m and its smooth models differ?

4.2. **The core map.**

Theorem 12. *Let X be a complex projective*[8] *manifold of dimension $n \geq 2$ and $c : X \to B$ the core map of X. If $p := \dim(B) \neq 1$ then $c_m : X_m \to B_m$ is (bimeromorphically) the core map of X_m.*

The case where B is a curve is studied in the next subsection (see also Remark 4.1).

Corollary 4.4. *If $n \geq 2$ and $p \neq 1$ then X is special if and only if so is X_m for some m.*

Indeed, X (resp. X_m) is special if and only if $\dim(B) = 0$ (resp. $\dim(B_m) = 0$), and $\dim(B_m) = m.\dim(B)$.

Proof of Theorem 12. Since the general fibres of c_m are products of special manifolds they are special (it is easy to see that a product of special manifolds is special). It is thus sufficient to show that the 'neat orbifold base' of c_m is of general type, knowing that so is the neat orbifold base of c. This requires some preliminary explanation.

Recall that $f : X \to B$ is neat if there exists a bimeromorphic map $u : X \to X_0$, with X_0 smooth, such that each f-exceptional divisor is also u-exceptional, and the complement of the open set $U = B \setminus D \subset B$ over which f is submersive is a snc divisor, as well as $f^{-1}(D) \subset X$. Such a neat model of $f_0 : X_0 \dashrightarrow B$ is

[8] The proof applies directly when X is compact Kähler.

obtained by flattening f_0, followed by suitable blow-ups. In this case, the support of D_f, the orbifold base of f, is snc too, and $\kappa(B, K_B + D_f)$ is minimal among all bimeromorphic models of f. More precisely, $\kappa(B, K_B + D_f) = \kappa(X, L_f)$, where $L_f := f^*(K_B)^{sat} \subset \Omega^p_X$, where $p := \dim(B)$, and $f^*(K_B)^{sat}$ is the saturation of $f^*(K_B)$ in Ω^p_X. See [Cam04] for details. Notice also that if $c : X \to B$ is a neat model of some $f_0 : X_0 \dashrightarrow B_0$, and if $x \in X$ is any point, there is another neat model $f' : X' \to B'$ dominating[9] $f : X \to B$ such that x does not belong to any f'-exceptional divisor on X', and lies in the image of the smooth locus of the reduction of a fibre of f'. If this condition is not realised on (X, f) it is then sufficient to suitably blow-up X, then flatten the resulting map by modifying B, and finally take a smooth model of the resulting f. The claim of Theorem 12 then holds true for (X, f) if it holds for (X', f').

Let $c : X \to B$ be neat with respect to $u : X \to X_0$, and let $c_m : X_m \to B_m$, together with a smooth model $c'_m : X'_m \to B'_m$ of c_m (i.e. X'_m, B'_m are smooth models of X_m, B_m).

Let us prove first that $c^m : X^m \to B^m$ is the core map of X^m, with orbifold base (B^m, D_{f^m}) and Kodaira dimension $m.\kappa(B, D_f)$. This follows inductively on m from the following easy lemma, which also shows that $D_{f^m} = \cup_{s \in S_m} s(D_f \times X^{m-1})$.

Lemma 4.5. *Let $f : X \to V, g : Y \to W$ be neat fibrations with orbifold bases $(V, D_f), (W, D_g)$. Then $f \times g : X \times Y \to V \times W$ is neat, its orbifold base is $(X \times Y, D_f \times W + V \times D_g)$, and its Kodaira dimension is $\kappa(V, D_f) + \kappa(W, D_g)$.*

Proof. If $E \subset V \times W$ is an irreducible divisor mapped surjectively on both V and W, there is only one irreducible divisor $F \subset X \times Y$ such that $(f \times g)(F) = E$, which has multiplicity 1 in $(f \times g)^*(E)$, since over $(v, w) \in E$ generic, $(f \times g)^{-1}(v, w) = X_v \times Y_w$, reduced. The other conclusions are obtained by a similar argument. □

• We now turn to the proof of Theorem 12. Let $c_m : X_m \to B_m$ be deduced by quotient from the core map c^m, and let $D_{c_m} \subset X_m$ be the direct image of D_{c^m} under the quotient map $q_B : B^m \to B_m$, so that $D_{c^m} = (q_B)^*(D_{c_m})$. It is sufficient to show that $\rho^*(c_m^*((K_{X_m} + D_{c_m})^{\otimes k})) \subset Sym^k(\Omega^{m.p}_{X'_m})$ for any (or some) $k > 0$ such that $k.(K_{X_m} + D_{c_m})$ is Cartier, where $\rho : X'_m \to X_m$ is a smooth model of X_m.

• If $p := \dim(B) = 0$, there is nothing to prove.

• We thus assume that $p := \dim(B) \geq 2$. The problem is local (in the analytic topology) on X^m, X_m, B^m, B_m. By the observations made above, we shall assume that the points $(x_1, \ldots, x_m)$ near which we treat the problem do not belong to any c-exceptional divisor, and are regular points of the reduction of the fibre of c containing them. The fibration c is thus given in suitable local coordinates on X and B by the map $c : (x_1, \ldots, x_n) \to (b_1, \ldots b_p)$ with $b_i := x_i^{t_i}, \forall i = 1, \ldots, p, p < n$, where the support of D_c is contained in the union of the coordinate hyperplanes $b_i = 0$ of B, the multiplicity of $b_i = 0$ in D_c being an integer t'_i, with $1 \leq t'_i \leq t_i, \forall i \leq p$, by the very definition of the orbifold base.

Since $c^*\Big(\Big(\frac{db_i}{b_i^{1-(1/t'_i)}}\Big)^{\otimes t'_i}\Big) = t_i^{t'_i}.x_i^{(t_i - t'_i)}.(dx_i)^{\otimes t'_i}$, we see that $(K_B + D_c)^{\otimes t}$ is Cartier and $c^*((K_B + D_c)^{\otimes t}) \subset Sym^t(\Omega^p_X)$, if $t = \mathrm{lcm}\{t'_i\}$.

Thus $(c^m)^*((K_{B^m} + D_{c^m})^{\otimes t}) \subset Sym^t(\Omega^{pm}_{X^m})$, this natural injection being deduced from the description of D_{c^m} given above (which shows that it is snc since so is

[9]In the sense that there exists birational maps $u' : X' \to X$ and $\beta' : B' \to B$ such that $f \circ u' = \beta' \circ f'$.

D_c). The saturation of the image of this injection inside $Sym^t(\Omega^{pm}_{X^m})$ is the line bundle generated by $T := (w_1\wedge\cdots\wedge w_m)^{\otimes t}$, where $w_j := dx_{1,j}\wedge\cdots\wedge dx_{p,j}, \forall j = 1,\dots,m$. Here $(x_{1,j},\dots,x_{n,j})$ are the local coordinates near the point $z_j \in X$, on the j-th component $X_j \cong X$ of X^m near the point $(z_1,\dots,z_m)$.

It is sufficient (considering separately the distinct points of the set $\{z_1,\dots,z_m\}$) to deal with the case where $z_j = z_k, \forall j,k \leq m$.

The operation of $\mathfrak{S}_m$ on the coordinates $x_{i,j}, i \leq n, j \leq m$ fixes the set of coordinates $x_{i,j}, i \leq p, j \leq m$ and induces on the vector space $\oplus_j V_j := \oplus_{i,j}\mathbb{C}.x_{i,j}, j \leq p$ they generate a representation which is a direct sum of p copies of the regular representation.

The conclusion then follows from Proposition 2.1. One checks the conditions[10] given in [Wei86] by using the (purely algebraic) proof of Prop.1, p. 1370, of [AA03], which says that if $\rho : \mathfrak{S}_m \to Gl(\oplus_{j=1}^{j=m}V)$ is a representation which is the direct sum of p copies of the regular representation, where V is a complex vector space of dimension $p \geq 2$, then $\sigma(g) = \frac{n}{2}.r.(\sum_{k=1}^{k=s}(r_k-1)) \geq r$, for any $g \in \mathfrak{S}_m$ which is the product of s non-trivial disjoint cycles of lengths r_k, and $r := \mathrm{lcm}((r_k)'s)$ is the order of g. Here $\sigma(g) := \sum_h a_h$, if the eigenvalues of $\rho(g)$ are ζ^{a_h}, where ζ is any complex primitive r-th root of the unity, and $0 \leq a_h < r$ for any h. □

4.3. **The core map of X_m when the base of c is a curve.** We now assume that $p := \dim(B) = 1$. Let $c : X \to B$ be the core map, and (B, D_c) its orbifold base. When $D_c = 0$, the situation is easy:

Theorem 13. *Assume that the core map $c : X \to B$ maps onto a curve B, and that its orbifold-base divisor $D_c = 0$. Then $c_m : X_m \to B_m$ is the core map if $m < g$, and X_m is special if $m \geq g$.*

Proof. Since $D_c = 0$, the fibration $c : X \to B$, and so c_m, has everywhere local sections, thus the same is true for c_m, and hence for any smooth birational model of c_m. The conclusion thus follows from the fact that B_m is of general type if $m < g$, and special if $m \geq g$. □

In the general case, we have a weaker statement:

Corollary 4.6. *If $c : X \to B$ is the core map, with B a curve, there is an integer $g(B, D_c) > 0$ such that X_m is special if $m \geq g(B, D_c)$. Moreover, X_m is not special if $m < g(B)$.*

Proof. By assumption, the orbifold curve (B, D_c) is of general type, hence 'good', meaning that there exists a finite Galois cover $h : \tilde{B} \to B$ which ramifies at order t' over each point $b \in D_c \subset B$, b of multiplicity t' in D_c. The normalisation $H : \tilde{X} \to X$ of the fibre-product $X \times_B \tilde{B}$ comes equipped with $\tilde{c} : \tilde{X} \to \tilde{B}$, which is its core map, since this fibration has everywhere local sections.

If $m \geq g(\tilde{B})$, then $\tilde{X}_m$, and so also X_m, is special. This shows the first claim.

The second claim follows from the fact that $B_{g(B)-1}$ is the Θ divisor on the Jacobian of B, and so it is of general type. If we now take $m \leq (g(B)-1)$, B_m is still of general type, as seen inductively on $m = 1,\dots,g(B)-2$ by contradiction, because the images of $\{a\} \times B_m, a \in B$ in B_{m+1} by the natural addition map are

[10] The simpler form of our tensor T reduces the conditions, for a given g, in the proof-not the statement-of Lemma 4 of [Wei86] to a single one: $\sigma(g) \geq r$ (in loc.cit the data ℓ, d, N, m correspond to t, pm, n, r here, respectively.)

injective and cover B_{m+1} when $a \in B$ varies. Since X_m fibres over B_m, we get that X_m is not special for $m \leq g(B) - 1$. □

Remark 4.7. It is possible to show a more precise result (not used here): if $\delta := deg(D_c)$, then X_m is special for $m \geq g(B) + \delta$, and non-special otherwise.

It is now easy to put all the previous results together to get Theorem 1 as a more synthetic statement.

Proof of Theorem 1. The direct implication follows from Corollary 4.4, while the converse implication is a consequence of Theorem 13, and Corollary 4.6. □

5. Dense entire curves in symmetric powers

5.1. **Dense entire curves in $Sym^m(G \times C)$.** Let G (resp. C) be a curve of genus $g(G) \leq 1$ (resp. $g := g(C) > 1$), and $S = G \times C$, then S_m is special if and only if $m \geq g$, which we assume from now on. Theorem 13 shows that S_m is 'special' (hence 'weakly-special'), while of course, S^m is not 'weakly special'. This section is devoted to the proof of Theorem 3: S_m contains (lots of) entire curves $h : \mathbb{C} \to S_m$ with dense (not only Zariski-dense) image if (and only if) $m \geq g$. Note indeed that if $m < g$, then S_m fibres over C_m by means of its core map, which implies that the entire curves on S_m are contained in the fibres.

The statement of Theorem 3 was suggested by Ariyan Javanpeykar as a test case for the conjecture by the second named author, that special manifolds should contain dense entire curves. The arithmetic counterpart were that S_m is 'potentially dense' if defined over a number field. Theorem 3 can be obtained as a consequence of the following more precise result:

Theorem 14. *If $S = G \times C$ is as above, the following are equivalent:*

1. $m \geq g$,
2. S_m is special,
3. S_m contains dense entire curves.

Proof. We shall assume here that $G = \mathbb{P}_1$, the proof when G is an elliptic curve being completely similar (just replacing $\mathbb{C} \subset \mathbb{P}_1$ by $\mathbb{C} \to G$ the universal cover). Observe that C_m contains dense entire curves, since it fibres surjectively over $\mathrm{Jac}(C)$ as a $\mathbb{P}^r$-bundle, with $r := m - g$, over the complement in $\mathrm{Jac}(C)$ of a Zariski-closed subset of codimension at least 2.

Take a dense entire curve $f : \mathbb{C} \to C_m$, let $V \subset \mathbb{C} \times C$ be the graph of the family of m-tuples of points of C parameterized by $\mathbb{C}$ via f (i.e. $V := \{w := (z, c) | c \in C, c \in f(z)\}$. The map $\pi : V \to \mathbb{C}$ sending $w = (z, c)$ to z is thus proper, open and of geometric generic degree m. In particular, V is a Stein curve (not necessarily irreducible). Let $F : V \to C$ be the projection on the second factor. Let $g : V \to \mathbb{C} \subset \mathbb{P}_1 = G$ be any holomorphic map. The product map $g \times F : V \to \mathbb{C} \times C \subset G \times C = S$ is thus well-defined. We now define the map $h : \mathbb{C} \to S_m$ by sending $z \in \mathbb{C}$ to the m-tuple of S defined by: $(g \times F)(\pi^{-1}(z)) \subset S$.

We now just need to check that the map $g : V \to \mathbb{C}$ can be chosen such that $h(\mathbb{C}) \subset S_m$ is dense there. Note first that if $(z_n)_{n>0}$ is a any discrete sequence of pairwise distinct complex numbers such that $\pi : V \to \mathbb{C}$ is unramified over each z_n, and if, for each $n > 0$, $(t_{n,1}, \ldots, t_{n,m})$ is an m-tuple of complex numbers, there exists a holomorphic map $g : V \to \mathbb{C}$ such that $g(w_{n,i}) = t_{n,i}, \forall n > 0, i = 1, \ldots, m,$

where $(w_{n,1} = (z_n, c_{n,1}), \dots, w_{n,m} = (z_n, c_{n,m})) = \pi^{-1}(z_n)$, and $(c_{n,1}, \dots, c_{n,m}) := f(z_n) \in C_m$ (the ordering being arbitrarily chosen).

It is now an elementary topological fact that the sequences $(t_{n,1}, \dots, t_{n,m}), n > 0$ can be chosen in such a way that the sequence $(s_{n,1}, \dots, s_{n,m})_{n>0} \in S^m$ is dense in S^m, where $s_{n,i} := (t_{n,i}, c_{n,i}) \in S, \forall n > 0, i = 1, \dots, m$. □

Remark 5.1. The preceding arguments work more generally for $X = G \times C$, when C, m are as above, but G enjoys the following property: for any smooth complex Stein curve $V \to \mathbb{C}$ proper over $\mathbb{C}$, and any sequence of distinct points $w_n \in W, t_n \in G$, there exists a holomorphic map $g : V \to G$ such that $g(w_n) = t_n, \forall n$.

This property is satisfied for G a complex torus or a unirational manifold. The same arguments would show the same result for G rationally connected if one could answer positively the following question, answered positively in [CW19], when $V = \mathbb{C}$:

Question: For $m, C, \pi : V \to \mathbb{C}$ defined as above, let $w_n \in V, t_n \in G, n \in \mathbb{Z}_{>0}$ be a sequence of points. Assume that the points $\pi(w_n) \in \mathbb{C}$ are all pairwise distinct. Does there exist a holomorphic map $g : V \to G$ such that $g(w_n) = t_n, \forall n$ if G is rationally connected?

Remark 5.2. Let now $\Delta^{(m)} \subset S^m$ be the 'small diagonal', consisting of m-tuple of points of which 2 at least coincide. Thus $(S^m)^* := S^m \setminus \Delta^{(m)}$ admits a surjective (but non-proper) map to C^m.

Let $\Delta_m \subset S_m$ be defined as: $\Delta_m := \mathfrak{D}_2(S_m) = q(\Delta^{(m)})$. We thus have, too: $\Delta^{(m)} = q^{-1}(\Delta_m)$. The restricted map $q : (S^m)^* \to (S_m)^* := S_m \setminus \Delta_m$ is thus proper and étale.

Let $d_{(S^m)^*} := d_{S^m|(S^m)^*}$ (by [Kob98]) be the Kobayashi pseudometric on $(S^m)^*$. Since the Kobayashi pseudometric on S^m is the inverse image of that on C^m by the natural projection $\gamma^m : S^m \to C^m$, any entire curve $h : \mathbb{C} \to S^m$ (and so even more in $(S^m)^*$) has to be contained in some fibre of γ^m. Moreover, the Kobayashi pseudometric on $(S_m)^*$ is comparable to its inverse image in $(S^m)^*$ (and can be explicitly described). This shows that any entire curve in $(S_m)^*$ is contained in the image by q of a fibre of γ_m, and is in particular algebraically degenerate (although there are lots of dense entire curves on S_m, none of these avoids Δ_m).

This gives a counterexample to an analytic version of the 'puncture problem' of [HT01], similar to the arithmetic one of [Lev].

5.2. **$\mathbb{C}^{2g}$-dominability of $S^{[g]}$, the g-th symmetric product of generic projective $K3$-surfaces.** Let S be a smooth projective $K3$-surface with[11] $Pic(S) \cong \mathbb{Z}$, generated by an ample line bundle L of degree $2(g-1), g > 1$. Such $K3$-surfaces are thus generic among projective $K3$-surfaces admitting a primitive ample line bundle of degree $2.(g-1)$.

The objective is to prove the following

Theorem 15. *For any such S, there is a (transcendental) meromorphic map $h : \mathbb{C}^{2g} \dashrightarrow S_g$ whose image contains a nonempty Zariski open subset U of S_g (we say that S_g is "$\mathbb{C}^{2g}$-dominable"). In particular, for any countable subset P of U, there is an entire curve on S_g whose image contains P. If P is dense in S_g, so is the image of this entire curve.*

[11]with some more work, it is probably possible to extend the next result to any projective $K3$-surface, by taking for L an ample and primitive line bundle with g minimal.

Remark 5.3. 1. The proof rests on a suitable abelian fibration $S_g \dashrightarrow \mathbb{P}^g$. Our result may thus be seen as analog to the case when S is an elliptic $K3$ surface (over $\mathbb{P}_1$) and $g = 1$, shown in [BL00].

2. Our result is analogous to the arithmetic situation treated by [HT01].

3. Since S_g is special, Theorem 15 solves in a stronger form one of the conjectures of [Cam04] in this particular case.

4. One may expect the conclusion of Theorem 15 to hold for $S^{[k]}$, any $k > 1$ and any $K3$-surface (projective or not).

Before starting the proof, we recall some of the objects which have been attached to such a pair (S, L).

The Hilbert scheme of g points: The Hilbert scheme $S^{[g]}$ of points of length g on S, equipped with the Hilbert to Chow birational morphism $\delta : S^{[g]} \to S_g$, is known to be smooth ([Fog68], Theorem 2.4) and holomorphically symplectic [Bea83]).

The Relative Jacobian: The line bundle L determines $\mathbb{P}(H^0(S, L))^*) := \mathbb{P}^g$, the g-dimensional projective space (by Riemann-Roch and Kodaira vanishing). The linear system $|L|$ is base-point free and the associated map $\varphi : S \to \mathbb{P}^g$ is an embedding for $g \geq 3$ (a double cover ramified over a sextic for $g = 2$). For each $t \in \mathbb{P}^g$, the corresponding zero locus of a non-zero section of $|L|$ is an irreducible and reduced (by the cyclicity of $Pic(S)$ assumption) curve of genus g denoted C_t. The incidence graph of this family of curves is denoted by $\gamma : \mathcal{C} \to \mathbb{P}^g$. For $d \in \mathbb{Z}$, the relative Jacobian fibration $j^d : J^d \to \mathbb{P}^g$ has fibre over t the Jacobian J^d_t of degree d line bundles on C_t. The Jacobian J^0_t of degree 0 line bundles on C_t (isomorphic to J^d_t by tensorising with any given line bundle of degree d) is a complex Hausdorff Lie group of dimension g quotient of $H^1(C_t, \mathcal{O}_{C_t})$ by the (closed) discrete subgroup $H^1(C_t, \mathbb{Z})$ ([BPVdV84, II.2, Proposition (2.)]). Thus, denoting with $j^0 : J^0 \to \mathbb{P}^g$ the relative Jacobian of degree 0 (instead of d) line bundles on the $C_t's$, and $V := R^1\gamma_*(\mathcal{O}_\mathcal{C}) \to \mathbb{P}^g$, this sheaf is locally free and thus a vector bundle $w : V \to \mathbb{P}^g$ of rank g on $\mathbb{P}^g$. By [Gro62, Theorème 3.1], the relative Picard scheme is separated[12], and so the relative discrete group $R^1\gamma_*(\mathbb{Z}) \to \mathbb{P}^g$ is closed in V. Taking the quotient, we get:

Lemma 5.4. *There is a holomorphic and surjective unramified map $H : V \to J^0$ over $\mathbb{P}^g$.*

The compactified Jacobian: For $d \in \mathbb{Z}$, this is the compactification $\bar{j}^d : \overline{J^d} \to \mathbb{P}^g$ of J^d over $\mathbb{P}^g$ obtained as a component of the moduli space of simple sheaves on S ([Muk84]). This variety is compact smooth, holomorphically symplectic and, for $d = g$, birational to $S^{[g]}$ ([Bea91, Proposition 3]). We denote with $\sigma : S^{[g]} \dashrightarrow \overline{J^g}$ this birational equivalence.

The covering by singular elliptic curves. By [BPVdV84, VIII, Theorem 23.1] (see references there for the original proofs), there is a nonempty curve in $\mathbb{P}^g$ parametrizing (singular) curves $C_t's$ with elliptic normalizations. This family (and each of its components) covers S. Choosing g generic (normalised) members $E_1, \dots, E_g$ of such an irreducible family provides a product $\varepsilon : E := E_1 \times \cdots \times E_g \subset S^g$. By [HT01, proof of Theorem 6.1], the composed projection $\bar{j}^g \circ \sigma \circ \varepsilon : E \to \mathbb{P}^g$ is a (meromorphic) multisection of the (meromorphic) fibration $\tau := (\bar{j}^g) \circ \sigma : S^{[g]} \to \mathbb{P}^g$.

[12]We thank D. Markushevich for this reference and helpful comments.

This fact is actually easy to prove, since if C_t is smooth, it cuts each of the $E_i's$ in finitely many distinct points, and so the intersection of E with C_t^g is finite, and surjective on the fibre of S_g over $\mathbb{P}^g$.

Proof. We can now prove Theorem 15. For any complex manifolds M, R equipped with a holomorphic map $\mu : M \to \mathbb{P}^g, r : R \to \mathbb{P}^g$, we denote with $R(M) := R \times_{\mathbb{P}^g} M$, equipped with the projections $\mu_M : R(M) \to M, r_M : R(M) \to R$. This, applied to $R = V, R = \overline{J^d}, R = S^{[g]}(M)$, gives the fibre products $V(M), \overline{J^d}(M), S^{[g]}(M)$.

We have two meromorphic and generically finite maps $\epsilon : E \dashrightarrow S^{[g]}$, and $\sigma \circ \epsilon : E \dashrightarrow \overline{J^g}$. Denote with E_t the fibre of E over $t \in \mathbb{P}^g$. We get a birational map $\beta : \overline{J^g}(E) \dashrightarrow \overline{J^0}(E)$ over E by sending a generic pair $(j, (e_1, \dots, e_g)_t) \in J_t^g \times E_t$ to $j \otimes \lambda^{-1}$, if $\lambda \in J_t^g$ is the line bundle on C_t with a nonzero section vanishing on the g points e_i.

Let $\pi : E' \to E$ be a modification making these maps holomorphic. Let $w_E : V(E') \to E'$ be the rank-g vector bundle on E' lifted from $w : V \to \mathbb{P}^g$. We get also a natural holomorphic map, unramified and surjective $H_E : V(E') \to J^0(E')$ over E'. Let $\mathcal{E} := \pi_*(V)$:this is a rank-g coherent sheaf on E, and there is a natural evaluation map: $\pi^*(\mathcal{E}) \to V$ over E'.

Let now $\rho : \overline{E} \to E$ be the universal cover, so that $\overline{E} \cong \mathbb{C}^g$. Let $\pi' : E' \times_E \overline{E} \to \overline{E}$ be deduced from $\pi : E' \to E$ by the base change ρ. Hence π' is a proper modification. The sheaf $\rho^*(\mathcal{E})$ on $\overline{E}$ is coherent, hence generated by its global sections since $\overline{E}$ is Stein. Let $W \subset H^0(\overline{E}, \rho^*(\mathcal{E}))$ be a vector subspace of dimension g which generates $\rho^*(\mathcal{E})$ at the generic point of $\overline{E}$, and let $ev : W \times \overline{E} \cong \mathbb{C}^{2g} \to V(E')$ be the resulting meromorphic and bimeromorphic map, obtained from the injection $\pi'^* : H^0(\overline{E}, \rho^*(\mathcal{E})) \to H^0(E' \times_E \overline{E}, V(E'))$.

We thus obtain a dominating meromorphic map $\mathbb{C}^{2g} \to S^{[g]}$ by composing ev with the bimeromorphic maps between $\overline{J^0}(E'), \overline{J^g}(E'), S^{[g]}(E')$, and finally projecting $S^{[g]}(E')$ onto $S^{[g]}$.

This completes the proof of Theorem 15. □

Part 2. Hyperbolicity of symmetric powers

6. A remark on the Kobayashi pseudometric

For any (irreducible) complex space Z, let d_Z be its Kobayashi pseudo-distance (see [Dem12, §1. A] for the proper definition). We say that Z is generically hyperbolic if d_Z is a metric on some nonempty Zariski open subset of Z.

Question 2. Assume X is smooth, compact and generically Kobayashi hyperbolic with $n > 1$. Is then X_m is generically Kobayashi hyperbolic for any $m > 0$?

Let us make one remark in this context. Let $(X^m)^* \subset X^m$ be the Zariski open subset consisting of ordered m-tuples of distinct points of X. The complement of $(X^m)^*$ has codimension $n \geq 2$ in X^m. By [Kob98, Theorem 3.2.22], we have $d_{X^m|(X^m)^*} = d_{(X^m)^*}$. Let $q_m : X^m \to X_m$ denote the quotient map, and $X_m^* := q_m((X^m)^*)$, so that X_m^* has a complement of codimension n in X_m as well, which is the singular set of X_m. Moreover, $(X^m)^* = q_m^{-1}(X_m^*)$. From [Kob98, 3.1.9 and 3.2.8], we get:

$$d_{X_m^*}([x_1, \dots, x_m], [y_1, \dots, y_m]) = \inf_{s \in S_m}\{\max_{i=1,\dots,m}\{d_X(x_i, y_{s(i)})\}\}.$$

Although the complement X_m^{sing} of X_m^* in X_m has codimension $n \geq 2$ (and the singularities are canonical quotient), it is not true that $d_{X_m|X_m^*} = d_{X_m^*}$ in general, as the following example shows. Even more, the pseudometric may degenerate away from X_m^{sing}, so the problem is not a local one near X_m^{sing}.

Example 2. Let $C \subset X$ be an irreducible curve of geometric genus g with normalisation $\hat{C}$ on X, and take $m \geq g$. Then $\hat{C}_m \to \mathrm{Alb}(C)$ is a surjective morphism with generic fibres $\mathbb{P}_{m-g}$, and there is then a natural generically injective map from $\hat{C}_m$ to X_m showing that d_{X_m} vanishes identically on its image.

If the answer to Question 2 is affirmative (as it should be if and only if X is of general type, after S. Lang's conjectures), the vanishing locus of d_{X_m} appears to have an involved structure. In particular, it should contain the union of all the $\hat{C})_m$ whenever $g(\hat{C}) \leq m$, and this union should not be Zariski dense.

Example 3. The simplest possible example might be a surface $S := C \times C'$, where C, C' are smooth projective curves of genus 2, and $m = 2$. In this case, the natural map $S_2 \to C_2 \times C_2'$ is a ramified cover of degree 2 branched over $R := (2C) \times C_2' \cup C_2 \times (2C')$, where $(2C) \subset C_2$ is the divisor of double points (and similarly for $(2C')$). Notice that C_2 identifies naturally with the $Pic_2(C)$, the Picard variety of line bundles of degree 2 on C, isomorphic to $Jac(C)$, blown-up over the point $\{K_C\}$, and $2C$ embeds C in C_2, its image meeting the exceptional divisor of C_2 in the 6 ramification points of the map $C \to \mathbb{P}_1$ given by the linear system $|K_C|$. Thus $2C \subset C_2$ is an ample divisor (similarly for C').

As a first step towards Question 2, let us show the following result which in particular implies that entire curves in the above example cannot be Zariski dense.

Proposition 6.1. *Let X be a complex projective variety of dimension n with irregularity $q := h^0(X, \Omega_X)$.*

(1) If $m \cdot n < q$ then entire curves in X_m are not Zariski dense.
(2) If X is of general type, $n \geq 2$ and $m \cdot n \leq q$ then entire curves in X_m are not Zariski dense.

Proof. Let $\alpha : X \to A$ be the Albanese map. It induces the Albanese map $\alpha_m : X_m \to A$. If $\dim X_m = m \cdot n < q = \dim A$ then by the classical Bloch-Ochiai's Theorem, entire curves in X_m are not Zariski dense. If X is of general type, by [AA03], X_m is of general type. Therefore by [Yam04, Corollary 3.1.14], if $\dim X_m = m \cdot n \leq q = \dim A$, entire curves in X_m are not Zariski dense. □

7. Jet differentials over symmetric powers

In this section, we will present our main criterion for hyperbolicity of symmetric powers X_m, in terms of the existence of jet differentials on X (see Theorem 16).

7.1. **Jet differentials on resolutions of quotient singularities.** We recall here some basic definitions related, on the one hand, to natural orbifold structures on resolution of quotients singularities (see [CRT19, Cad18, CDG19]), and on the other hand, to orbifold jet differentials (see [CDR18]). The basic result we will need is given by Proposition 7.1.

7.1.1. *Jet differentials on orbifolds.* Let us give some details about the very basic notion of orbifold jet differentials that we will use in the following. For our purposes, it will be enough to consider only orbifolds of the form $(X, \Delta = \sum_i (1 - \frac{1}{m_i}) D_i)$, with $m_i \in \mathbb{N}_{\geq 1}$. Also, rather than using the *geometric orbifold jet differentials* defined in [CDR18], it will also suffice to consider jet differentials adapted to *divisible holomorphic curves* in the sense of [*loc. cit.*, Definition 1.1]. The latter jet differentials admit a very simple description. For any $k, r \in \mathbb{N}$, we will denote by $E_{k,r}^{GG} \Omega_{(X,\Delta)}^{\mathrm{div}}$ the vector bundle of divisible orbifold jet differentials of order k and degree r, whose sections in orbifold local charts adapted to Δ can be described as follows. Assume that $(t_1, ..., t_p, t_{p+1}, ..., t_n) \in U \longmapsto (t_1^{m_1}, ..., t_p^{m_p}, t_{p+1}, ..., t_n) \in V$ is such a chart. Then, the local sections of $E_{k,r}^{GG} \Omega_{(X,\Delta)}^{\mathrm{div}}$ corresponds to the regular sections of $E_{k,r}^{GG} \Omega_U$ on U, which are invariant under the deck transform group. Remark that we could also have defined $E_{k,r}^{GG} \Omega_{(X,\Delta)}^{\mathrm{div}}$ in terms of a global *adapted covering* instead of local orbifold charts.

7.1.2. *Natural orbifold structure on resolutions of a quotient singularity.* Consider now a quotient $Y = {}_G\backslash^X$ where X is smooth, and G finite. If $\widetilde{Y} \longrightarrow Y$ is a resolution of singularities, we can endow it with a natural orbifold structure, by assigning to every exceptional divisor $E \subset \widetilde{Y}$ the rational multiplicity $1 - \frac{1}{m}$, where m is the order of the element $\gamma \in G$ associated with the meridional loop around the generic point of E (see [CDG19, Cad18]).

With this notation, the following proposition is then essentially tautological.

Proposition 7.1. *Let X be a complex manifold, and let $G \subset \mathrm{Aut}(X)$ be a finite subgroup. Let $p : X \longrightarrow Y = {}_G\backslash^X$ be the quotient map, and $\widetilde{Y} \stackrel{\pi}{\longrightarrow} Y$ be a resolution of singularities. Let $(\widetilde{Y}, \Delta)$ be the natural orbifold structure on $\widetilde{Y}$. Let A be a G-invariant divisor on X, and B the associated Cartier divisor on Y such that $p^* B = A$.*

For $k, r \in \mathbb{N}$, we let $\sigma \in H^0(X, E_{k,r}^{GG} \Omega_X \otimes \mathcal{O}(-A))$ be a G-invariant section. Then $\pi^* p_* \sigma$ induces an element of $H^0(\widetilde{Y}, E_{k,r}^{GG} \Omega_{(\widetilde{Y},\Delta)}^{\mathrm{div}} \otimes \mathcal{O}(-\pi^* B))$.

Remark 7.2. With the notation of the previous proposition, we see that if r is divisible enough, and if f is a local section of $\mathcal{O}_{\widetilde{Y}}(-r\Delta) \subset \mathcal{O}_{\widetilde{Y}}$, then $f \cdot \pi^* p_* \sigma$ is a holomorphic section of $E_{k,r}^{GG} \Omega_{\widetilde{Y}} \otimes \mathcal{O}(-\pi^* B)$.

7.2. **A first criterion for the hyperbolicity of symmetric powers.** Before presenting our next hyperbolicity result, let us first prove a proposition that will allow us later on to compensate for the divergence of natural orbifold objects on resolutions of X_m. We resume the notation introduced in Section 2.1.

Proposition 7.3. *Let X be a complex projective manifold, and let A be a very ample divisor on X. Let $\pi : \widetilde{X}_m \longrightarrow X_m$ be a log-resolution of singularities, and let Δ be the exceptional divisor with its reduced structure. Then*

$$\mathbb{B}(\pi^* A_\flat - \frac{1}{2(m-1)} \Delta) \subset |\Delta|,$$

where $\mathbb{B}$ denotes the stable base locus.

We break the proof of this proposition into several lemmas.

Lemma 7.4. *Let U be a complex manifold, let $G \subset \mathrm{Aut}(U)$ be a finite group, and let $p : U \longrightarrow {}_G\backslash^{U} = V$ be the quotient map. Let A be a divisor on X, and let $A^\sharp = \sum_{\gamma \in G} \gamma^* A$. Note that $A^\sharp$ is G-invariant, so there exists a Cartier effective divisor $A_\flat$ on V such that $p^* A_\flat = A^\sharp$. Let $W \subset U$ be an irreducible component of the subset of points stabilized by some element of G. Let $s \in \Gamma(U, A^\sharp)$ be a G-invariant section vanishing at order r along W, for some $r \geq 1$. Then, we have the following.*

(1) s descends to a section $\sigma \in \Gamma(V, A_\flat)$;
(2) let $\widetilde{X} \xrightarrow{\pi} X$ be a resolution of singularities, and let $E \subset \widetilde{X}$ be an exceptional divisor such that $\pi(E) \subset p(W)$. Let m be the multiplicity of E for the natural orbifold structure on $\widetilde{X}$. Then, $\pi^*\sigma$, seen as a section of $\pi^* A_\flat$, vanishes at order $\geq \frac{r}{m}$ along E.

Proof. (1) is trivial, by definition of $A_\flat$. Let us prove (2). Let $H \subset G$ be the stabilizer of the generic point of $\pi(E)$. By definition of $A^\sharp$, we may find an H-invariant trivialization e of $A^\sharp$ near this generic point. Besides, $s = f\, e$ for some H-invariant holomorphic function f vanishing at order r along W. Consider a polydisk $D \cong \Delta^n$ centered around a generic point of E, and let D' be the normalization of the fibered product of D and U over V. We obtain the following diagram:

$$\begin{array}{ccccc} & & D' \cong \Delta \times \Delta^{n-1} & \xrightarrow{\pi'} & U \\ & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ (\Delta^n \cap E) = \{0\} \times \Delta^{n-1} & \hookrightarrow & D \cong \Delta \times \Delta^{n-1} & \xrightarrow{\pi} & V \end{array}$$

Since f is H-invariant, $f \circ \pi' = f' \circ p'$ for some holomorphic function f' on $D \cong \Delta \times \Delta^{n-1}$. Moreover, we have $\sigma = f'\, e_\flat$, where $e_\flat$ is the section of $A_\flat$ induced by e. The holomorphic function f vanishes at order $r > 0$ along V, so $f \circ \pi'$ vanishes at order $\geq r$ along $\{0\} \times \Delta^{n-1}$. Since $p'(w, z) = (w^m, z)$, this implies that f' vanishes at order $\geq \frac{r}{m} > 0$ along $\{0\} \times \Delta^{n-1} \subset \Delta^n$. This ends the proof. □

Lemma 7.5. *Let $N, m \geq 1$. We define $V = \mathbb{P}^N \times ... \times \mathbb{P}^N$ to be a product of m copies of $\mathbb{P}^N$. Let $\mathfrak{D} = \{(z_1, ..., z_m) \in V \mid \exists i \neq j,\ x_i = x_j\} \subset V$ be the diagonal locus. Let $A \subset \mathbb{P}^N$ be a hyperplane section, and let $A^\sharp = \sum_{i=1}^m \mathrm{pr}_i^* A$.*

Then, for any $z \in V \setminus \mathfrak{D}$, there exists a $\mathfrak{S}_m$-invariant section

$$s \in \Gamma(V, \mathcal{O}_V(2(m-1)A^\sharp)),$$

with $s(x) \neq 0$, and such that s vanishes at order 2 along $\mathfrak{D}$.

Proof. Let $z = (z_1, ..., z_m) \in V \setminus \mathfrak{D}$. Write $(\mathbb{P}^N)_i$ to denote the i-th factor of V. For any $i < j$, we have $z_i \neq z_j$, so for two generic hyperplane linear sections $X, Y \in |A|$, we have

$$X(z_i)Y(z_j) - X(z_j)Y(z_i) \neq 0. \tag{1}$$

Indeed, we can choose X, Y so that $X(z_i) \neq 0$ and $X(z_j) = 0$ (resp. $Y(z_i) = 0$ and $Y(z_j) \neq 0$).

Now, choose two generic linear sections X, Y in $|A|$, and for each $1 \le i \le m$, let X_i and Y_i be the corresponding section on the copy $(\mathbb{P}^N)_i$. We let

$$s = \prod_{i<j} (X_i Y_j - X_j Y_i)^2$$

This is a section of $\bigotimes_{i=1}^m p_i^* \mathcal{O}(2(m-1)) = \mathcal{O}(2(m-1)A^\sharp)$. By the argument above, we can pick s so that $s(z) \neq 0$, and s vanishes on $\mathfrak{D}$ at order 2 by Lemma 7.6. We check that s is invariant under all transpositions $(i\,j) \in \mathfrak{S}_m$. This proves that s is $\mathfrak{S}_m$-invariant. □

Lemma 7.6. *Let X_1, Y_1 be two generic hyperplane sections on $\mathbb{P}^N$, and let X_2, Y_2 denote the same sections on a second copy of $\mathbb{P}^N$. Then the homogeneous polynomial $X_1 Y_2 - X_2 Y_1$ vanishes at order 1 along the diagonal of $\mathbb{P}^N \times \mathbb{P}^N$.*

Proof. We let $2u = X_1 + X_2$, $2v = X_1 - X_2$ (resp. $2u' = Y_1 + Y_2$, $2v' = Y_1 - Y_2$). Then, we can write

$$\begin{aligned} X_1 Y_2 - X_2 Y_1 &= (u+v)(u'-v') - (u-v)(u'+v') \\ &= -2uv' + 2u'v. \end{aligned}$$

This expression is of degree 1 in v' and v, so for generic u, u', it vanishes at order one along the diagonal. □

The proof of Proposition 7.3 is now straightforward.

Proof of Proposition 7.3. Let $x \in \widetilde{X}_m \setminus |\Delta|$, and let $x_0 \in X^m$ be such that $p(x_0) = \pi(x)$. Since x is not in $|\Delta|$, x_0 is not in the diagonal locus of X^m. Using the embedding $X \subset \mathbb{P}^N$ provided by the very ample divisor A, Lemma 7.5 gives a $\mathfrak{S}_m$-invariant section $s \in H^0(X^m, 2(m-1)A^\sharp)$ such that $s(x_0) \neq 0$, and such that s vanishes at order 2 along the diagonal locus.

We may now see s as a a section σ of $2(m-1)A_\flat$. Applying Lemma 7.4 to s, we see that the induced section

$$\pi^* \sigma \in H^0(\widetilde{X}_m, 2(m-1)\pi^* A_\flat)$$

vanishes along $|\Delta|$. Moreover, we have $\pi^*\sigma(x) \neq 0$, which gives the result. □

We are ready to state our hyperbolicity criterion (announced in Theorem 4), in terms of the existence of sufficiently many jet differentials of bounded order on X. Again, we refer to [Dem12] for the basic definitions related to jet differentials. Let us simply recall that the locus of singular jets $X_k^{GG,\mathrm{sing}} \subset X_k^{GG}$ is the subset of all classes of k-jets $[f : \Delta \to X]_k$ such that $f'(0) = 0$. Also, if $V \subset H^0(X, E_{k,r}^{GG}\Omega_X)$ is a vector subspace, then $\mathrm{Bs}(V) \subset X_k^{GG}$ is the subsets of classes of the k-jets which are solutions to every equation in V.

Theorem 16. *Let X be a complex projective manifold. Let A be a very ample line bundle on X. Let $Z \subset X$, and $k, r, d \in \mathbb{N}^*$. We make the following hypotheses.*

(1) Assume that

$$\mathrm{Bs}\left(H^0(X, E_{k,r}^{GG}\Omega_X \otimes \mathcal{O}(-dA))\right) \subset X_k^{GG,\mathrm{sing}} \cup \pi_k^{-1}(Z).$$

(2) Assume that $\frac{d}{r} > 2m(m-1)$.

Then, $\mathrm{Exc}(\widetilde{X}_m) \subset |\Delta| \cup \pi^{-1}(\mathfrak{d}_1(Z))$.

Proof. Let $f : \mathbb{C} \longrightarrow \widetilde{X}_m$ be an entire curve such that $f(\mathbb{C}) \not\subset |\Delta|$. Let $U = \mathbb{C} - f^{-1}(|\Delta|)$, and, as before $\mathfrak{D} = \bigcup_{i \neq j} \{x_i = x_j\} \subset X^m$. We consider the following diagram:

$$\begin{array}{ccccc} \widetilde{U} & \xrightarrow{g} & X^m \setminus \mathfrak{D} & \xrightarrow{\mathrm{pr}_i} & X \\ \downarrow q & & \downarrow p & & \\ U & \xrightarrow{f} & (X_m)_{\mathrm{reg}} & & \end{array}$$

where q is the universal covering map, and g is an arbitrary lift of f. Without loss of generality, we can assume that all $\mathrm{pr}_i \circ g$ are non-constant $(1 \leq i \leq m)$. Indeed, if one of these maps is constant, it suffices to replace X^m (resp. X_m) by the product $Y = X \times ... \times X$ over a number $m' < m$ of factors (resp. by $X_{m'} = {\mathfrak{S}_{m'}} \backslash Y$).

We may assume that $\mathrm{Im}(\mathrm{pr}_i \circ g) \not\subset Z$ for all $1 \leq i \leq m$, otherwise the proof is finished. Thus, there exists $t \in \widetilde{U}$ such that $(\mathrm{pr}_i \circ g)(t) \notin Z$, and $(\mathrm{pr}_i \circ g)'(t) \neq 0$ for all $1 \leq i \leq m$. By the hypothesis (1), there exists $\sigma \in H^0(X, E^{GG}_{k,m}\Omega_X \otimes \mathcal{O}(-dA))$ such that for all $1 \leq i \leq m$, we have $\sigma_{g(t)} \cdot (\mathrm{pr}_i \circ g) \neq 0$, and in particular

$$\sigma(\mathrm{pr}_i \circ g) \not\equiv 0$$

for all i.

Thus, $\sigma^\sharp \overset{\text{def}}{=} \bigotimes_{i=1}^m \mathrm{pr}_i^*(\sigma)$ is a $\mathfrak{S}_m$-invariant jet differential in $H^0(X^m, E^{GG}_{k,rm}\Omega_X \otimes \mathcal{O}(-dA^\sharp))$ such that $\sigma^\sharp(g) \not\equiv 0$. By Proposition 7.1, $\sigma^\sharp$ induces a section

$$\sigma_\flat \in H^0(\widetilde{X}_m, E^{GG}_{k,rm}\Omega^{\mathrm{div}}_{(\widetilde{X}_m,\Delta)} \otimes \mathcal{O}(-d\pi^* A_\flat)).$$

We have moreover $\sigma_\flat(f) \not\equiv 0$.

Now, by Proposition 7.3, for $a \geq 1$ divisible enough, there exists $s \in H^0(\widetilde{X}_m, a(\pi^* A_\flat - \frac{1}{2(m-1)}\Delta))$ such that $s|_{f(\mathbb{C})} \not\equiv 0$. Thus, by the remark following Proposition 7.1, $s^{2rm(m-1)}\sigma_\flat^a$ induces a non-orbifold section

$$\sigma' \in H^0\left(\widetilde{X}_m, E^{GG}_{k,arm}\Omega_{\widetilde{X}_m} \otimes \mathcal{O}\left(a(2rm(m-1) - d)\pi^* A_\flat\right)\right),$$

and $\sigma'(f) \not\equiv 0$.

Since $A^\sharp$ is ample, and p is finite, the divisor $A_\flat$ is ample, so $\pi^* A_\flat$ is big on $\widetilde{X}_m$. But now, since $2rm(m-1) < d$, the existence of σ' is absurd by the fundamental vanishing theorem of Demailly-Siu-Yeung (see [Dem12]). □

7.3. **Applications.**

7.3.1. *Hypersurfaces of large degree.* Using Theorem 16, we can now obtain hyperbolicity results for the varieties X_m when $X \subset \mathbb{P}^{n+1}$ is a generic hypersurface of large degree. To do this, we will make use of several important recent results concerning the base loci of jet differentials on such hypersurfaces. Let us begin with the algebraic degeneracy of entire curves.

The recent work of Bérczi and Kirwan [BK19] gives new effective degrees for which a generic hypersurface has enough jet differentials to ensure the degeneracy of entire curves. This improvement of [DMR10] yields the following result.

Theorem 17 ([BK19])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree*

$$d \geq 16n^5(5n+4).$$

Then, if $r \gg 0$ is divisible enough, we have

$$\mathrm{Bs}\left[H^0(X, E^{GG}_{n,r}\Omega_X \otimes \mathcal{O}(-r\frac{d-n-2}{16n^5} + r(5n+3)))\right] \subset X_k^{GG,\mathrm{sing}} \cup \pi_k^{-1}(Z) \tag{2}$$

for some algebraic subset $Z \subsetneq X$.

Remark 7.7. As explained in [BK19], the coefficient $5n+3$ comes from Darondeau's improvements [Dar16] for the pole order of slanted vector fields on the universal hypersurface. It seems to us by reading [Dar16] that we should actually expect the slightly better value $5n-2$.

We deduce immediately from Theorem 16 the following consequence of this result.

Corollary 7.8. *Let $m, n \in \mathbb{N}^*$. Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree*

$$d \geq 16n^5(5n + 2m^2 + 4).$$

Then there exists $Z \subsetneq X$ such that $\mathrm{Exc}(X_m) \subset \mathfrak{d}_1(Z)$.

Proof. Because of (2), the conditions of Theorem 16 will be satisfied if

$$\left(\frac{d-n-2}{16n^5} - (5n+3)\right) > 2m(m-1),$$

which is implied by our hypothesis. We have then $\mathrm{Exc}(X_m) \subset (X_m)_{\mathrm{sing}} \cup \mathfrak{d}_1(Z)$ for some $Z \subsetneq X$. Since $(X_m)_{\mathrm{sing}}$ is a union of $X_{m'}$ for $m' < m$, an induction on m permits to conclude. □

It is also possible to obtain the hyperbolicity of X_m when X has large enough degree, using all the recent work around the Kobayashi conjecture (cf. [Bro17, Den17, Dem18, RY18]). The main result of [RY18] permits to reduce the proof of the hyperbolicity of X to results such as Theorem 17, and gives in particular the following.

Theorem 18 ([RY18])**.** *Let $d, n, c, p \in \mathbb{N}$. Suppose that for a generic hypersurface $X' \subset \mathbb{P}^{n+1+p}$ of degree d, we have*

$$\mathrm{Bs}\left(H^0(X', E^{GG}_{k,r}\Omega_{X'} \otimes \mathcal{O}(-1))\right) \subset X'^{GG,\mathrm{sing}}_k \cup \pi_k^{-1}(Z'),$$

for some algebraic subset $Z' \subset X'$ satisfying $\mathrm{codim}(Z') \geq c$. Then, for a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d, we have

$$\mathrm{Bs}\left(H^0(X, E^{GG}_{k,r}\Omega_X \otimes \mathcal{O}(-1))\right) \subset X_k^{GG,\mathrm{sing}} \cup \pi_k^{-1}(Z),$$

for some subset $Z \subset X$ with $\mathrm{codim}(Z) \geq c+p$.

Letting $d = n-1$, we can give a proof of Theorem 5 as a corollary of Theorem 18 and Theorem 16, combined with Theorem 17:

Corollary 7.9. *Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree*

$$d \geq (2n-1)^5(2m^2 + 10n - 1).$$

Then X_m is hyperbolic.

7.3.2. *Complete intersections of large degree.* We can also obtain a hyperbolicity result for symmetric products of generic complete intersections of large multidegree, using the work of Brotbek-Darondeau and Xie on Debarre's conjecture (see [BD18, Xie18]). The effective bound in the theorem below is provided by [Xie18].

Theorem 19 ([BD18, Xie18]). *Let $n, n', d \geq 1$, and assume that $n' \geq n$. Let $X \subset \mathbb{P}^{n+n'}$ be a complete intersection of multidegrees*

$$d_1, ..., d_{n'} \geq (n+n')^{(n+n')^2} \cdot d$$

Then $\Omega_X \otimes \mathcal{O}(-d)$ is ample. In particular

$$\mathrm{Bs}(H^0(X, E_{1,r}^{GG} \otimes \mathcal{O}(-rd)) = \emptyset$$

for $r \gg 1$.

By Theorem 16 and the same induction argument on m as above, the following corollary is immediate.

Corollary 7.10. *Let $m, n \in \mathbb{N}^*$ and let $n' \geq n$. Let $X \subset \mathbb{P}^{n+n'}$ be a generic complete intersection of multidegrees*

$$d_1, ..., d_{n_1} > (n+n')^{(n+n')^2}(2m(m-1))$$

Then X_m is hyperbolic.

Remark 7.11. For d_1 large enough, Corollary 7.10 is trivially implied by Corollary 7.9. Indeed, if $X \subset H$, where H is a degree d_1 hypersurface, X_m embeds in H_m.

8. Higher dimensional subvarieties

In this section, we gather several results related to the subvarieties of X_m, when X is a "sufficiently hyperbolic" manifold. In particular, when Ω_X is ample, we will show that a generic subvariety of X_m of codimension higher than $n-1$ is of general type (see Theorem 20).

Lemma 8.1. *Assume that X is a complex manifold of dimension n, with $n \geq 2$, and let $\mathfrak{S}_m$ act on X^m. Let $\alpha \in [0,1]$. If*

$$d \geq n(m-1) + 2 - \alpha \frac{(n-2)(m-2)}{2},$$

then the condition $(I'_{x,d,\alpha})$ of Section 2.2 is satisfied for every $x \in X^m$. In particular, if $d \geq n(m-1)+2$, then the condition $(I_{x,d})$ is satisfied for any $x \in X^m$.

Proof. Let $\sigma \in \mathfrak{S}_m \backslash \{1\}$, and let $\sigma = \sigma_1 ... \sigma_t$ be a decomposition of σ into cycles with disjoint supports. For each σ_i, let $r_i = \mathrm{ord}(\sigma_i)$, and assume that $r_1 \geq ... \geq r_l > 1$, and $r_{l+1} = ... = r_{l+s} = 1$, with $s = t - l$. Then, the order of σ is $r = \mathrm{lcm}(r_1, ..., r_l)$, and the a_i appearing in condition $(I_{x,d})$ are the integers $j\frac{r}{r_k}$ $(1 \leq k \leq s, 0 \leq j < r_k)$, each one repeated n times. We see in particular that 0 appears with multiplicity $nt = n(s+l)$, and that each non-zero a_i is larger than $\frac{r}{\max_{1 \leq j \leq l} r_j}$.

We need to check that for any choice of d distinct elements $a_{i_1}, ..., a_{i_d}$ among the a_i, the sum is larger than $(1-\alpha)r$. The lowest possible sum is reached when all the 0 appear in it. Thus, the sum of the a_{i_j} is larger than

$$(d - n(s+l)) \frac{r}{\max_{1 \leq j \leq l} r_j}.$$

The last quantity is larger than $r(1-\alpha)$ if the following inequality is satisfied:

$$n(s+l)+(1-\alpha)\max_{1\le j\le l} r_j \le d \tag{3}$$

Now, we have $\max\limits_{1\le j\le l} r_j \le \sum\limits_{1\le j\le l} r_j = m-s$, and $2l+s \le \sum_{1\le j\le l} r_j + s = m$ hence $l \le \frac{m-s}{2}$. Putting everything together, we see that the following is always satisfied:

$$n(s+l)+\max_{1\le j\le l} r_j \le \left(\frac{n}{2}+1\right)m + (1-\alpha)\left(\frac{n}{2}-1\right)s.$$

Since $n \ge 2$ and $1-\alpha \ge 0$, the right hand side is maximal if s is maximal, equal to $m-2$; this right hand side is then equal to $n(m-1)+2-\alpha\dfrac{(n-2)(m-2)}{2}$ (thus the maximum is reached for $r_1 = 2, r_2 = \ldots = r_t = 1$, i.e. when σ is a transposition). Thus, if $d \ge n(m-1)+2-\alpha\dfrac{(n-2)(m-2)}{2}$, the inequality (3) is satisfied, which gives the result. □

In the next definition, we state a condition that will later imply that a generic subvariety of X_m of high enough dimension is of general type (see Theorem 20).

Definition 8.2. Let X be a complex projective manifold, let $\Sigma \subsetneq X$ be a proper algebraic subset, and let A be an effective divisor on X. We say that X *satisfies the property* $(H_{\Sigma,A})$, if the following holds.

Let $V \subset X$ be a subvariety of arbitrary dimension d, not included in Σ and A. Then, there exists $q, r \ge 1$, and a section $\sigma \in H^0(X, (\bigwedge^d \Omega_X)^{\otimes q})$, with *non-zero* restriction

$$\sigma|_{(\bigwedge^d T_{V^{\mathrm{reg}}})^{\otimes q}} \in H^0(V^{\mathrm{reg}}, (\bigwedge^d \Omega_V)^{\otimes q} \otimes \mathcal{O}(-rA|_V)) - \{0\}.$$

Under suitable positivity hypotheses on the cotangent bundle of a complex manifold, it is not hard to check that the previous condition is satisfied, as we will show in the next proposition.

Recall that if $E \longrightarrow X$ is a vector bundle, its *augmented base locus* is the algebraic subset $\mathbb{B}_+(E) \subset X$ defined as follows. Let $p : \mathbb{P}(E) \longrightarrow X$ be projectivized bundle of rank one quotients of E, and $\mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}(E)$. Then, if A is any ample line bundle on X, we let

$$\mathbb{B}_+(E) = p(\mathbb{B}_+(\mathcal{O}(1))),$$

where $\mathbb{B}_+(\mathcal{O}(1)) = \bigcap\limits_{l\ge 1} \mathrm{Bs}(\mathcal{O}(l) \otimes p^*A^{-1})$. The *ample locus* of E is the (possibly empty) open subset $X \setminus \mathbb{B}_+(E)$.

Proposition 8.3. *Let X be a complex projective manifold such that Ω_X is big. Let A be any very ample divisor on X.*

(1) if $\mathbb{B}_+(\Omega_X) \neq X$, then X satisfies the property $(H_{\mathbb{B}_+(\Omega_X),A})$;
(2) if Ω_X is ample, then X satisfies the property $(H_{\emptyset,A})$.

Proof. (1) Let $V \subset X$ be a d-dimensional subvariety such that $V \not\subset \mathbb{B}_+(\Omega_X)$ and $V \not\subset A$. By general properties of ampleness of vector bundles, we have the inclusion $\mathbb{B}_+(\bigwedge^d \Omega_X) \subset \mathbb{B}_+(\Omega_X)$ (this can be seen easily e.g. from [Laz04, Corollary 6.1.16])

Thus, if $x \in V \setminus \mathbb{B}_+(\bigwedge^d \Omega_X)$ is a smooth point of V, and $w = \bigwedge^d T_{V,x}$, there exists $\sigma \in H^0(X, S^m(\bigwedge^d \Omega_X) \otimes \mathcal{O}(-A))$ such that $\sigma_x(w^{\otimes m}) \neq 0$. In particular, since σ vanishes along A, the restriction $\sigma|_V$ vanishes along $A \cap V$. The section σ satisfies our requirements.

(2) If Ω_X is ample, we have $\mathbb{B}_+(\Omega_X) = \emptyset$, so the result comes from the first point. □

In the next proposition, we show that the property $(H_{\Sigma,A})$ is stable under products.

Proposition 8.4. *Let X_i $(i = 1, 2)$ be complex projective manifolds, and denote by $p_1, p_2 : X_1 \times X_2 \longrightarrow X$ the canonical projections. Assume that each X_i satisfies the property (H_{Σ_i,A_i}) for some subvariety $\Sigma_i \subsetneq X_i$ and some divisor A_i on X.*

Then $X_1 \times X_2$ satisfies the property $(H_{\Sigma,A})$, where $\Sigma = p_1^{-1}(\Sigma_1) \cup p_2^{-1}(\Sigma_2)$, and $A = p_1^*A_1 + p_2^*A_2$.

Proof. Let $V \subset X_1 \times X_2$ be a d-dimensional subvariety such that $V \not\subset \Sigma$. Let $d_2 = \dim p_2(V)$, and let d_1 be the dimension of the generic fiber of $p_2 : V \longrightarrow p_2(V)$. We have $d_1 + d_2 = d$.

(1) We deal first with the case $d_2 = 0$. Then, we have $\dim p_1(V) = d$, and $p_1(V) \not\subset \Sigma_1$ because $V \not\subset \Sigma$. Since X_1 satisfies (H_{Σ_1}), there exists integers $q, r \geq 1$, and a section $\sigma \in H^0(X_1, (\bigwedge^d \Omega_{X_1})^{\otimes q})$ such that $\sigma|_{\bigwedge^d T_{p_1(V)^{\mathrm{reg}}}}$ vanishes at order r along A_1. Thus, $(p_1)^*\sigma \in H^0(X_1 \times X_2, (\bigwedge^d \Omega_{X_1})^{\otimes q})$. We also have $(p_1)^*\sigma|_{\bigwedge^d T_{V^{\mathrm{reg}}}} \not\equiv 0$, and this section vanishes at order r along $p_1^*A_1 + p_2^*A_2|_V = p_1^*A_1|_V$. This ends the proof in this case.

(2) Assume now that $d_2 > 0$. Let $x_2 \in X_2$ be generic so that $\dim(V_{x_2}) = d_1$ and $p_1(V_{x_2}) \not\subset \Sigma_1$, where $V_{x_2} = p_2^{-1}(x_2) \cap V$. Let $V_2 = p_2(V)$, and $V_1 = p_1(V_{x_2})$.

For each i, we have $V_i \not\subset \Sigma_i$, so there exists integers $q_i, r_i \geq 1$, and a section $\sigma_i \in H^0(X_i, (\bigwedge^{d_i} \Omega_{X_i})^{\otimes d_i})$ whose restriction to $(\bigwedge^{d_i} T_{V_i^{\mathrm{reg}}})^{\otimes q_i}$ vanishes at order r_i along $A_i|_{V_i}$. Then,

$$\sigma = (p_1^*\sigma_1)^{\otimes q_2} \otimes (p_2^*\sigma_2)^{\otimes q_1}$$

can be identified to a section in $H^0(X_1 \times X_2, (\bigwedge^{d_1} p_1^*\Omega_{X_1} \otimes \bigwedge^{d_2} p_2^*\Omega_{X_2})^{\otimes q_1 q_2})$. Since $\bigwedge^{d_1} p_1^*\Omega_{X_1} \otimes \bigwedge^{d_2} p_2^*\Omega_{X_2}$ is a direct factor of $\bigwedge^d \Omega_X \cong \bigwedge^{d_1+d_2}(p_1^*\Omega_{X_1} \oplus p_2^*\Omega_{X_2})$, we have obtained a section $\sigma \in H^0(X_1 \times X_2, (\bigwedge^d \Omega_{X_1 \times X_2})^{\otimes q_1 q_2})$ which does not vanish along V.

Moreover, by construction, the restriction of σ to $(\bigwedge^d T_{V^{\mathrm{reg}}})^{\otimes q_1 q_2}$ vanishes along $B|_V$, where $B = q_2 r_1\, p_1^*A_1 + q_1 r_2\, p_2^*A_2$. Since $q_2 r_1, q_1 r_2 > 0$, this restriction vanishes along A. This gives the result. □

In the case where $X_1 = X_2$, it is not hard to strengthen the property (H_Σ) to obtain sections σ invariant by permutation of X_1 and X_2. More precisely, we have the following:

Proposition 8.5. *Let X be a complex projective manifold satisfying the property $(H_{\Sigma,A})$ for some $\Sigma \subsetneq X$ and some ample divisor A on X. Let $\Sigma' \subset X^m$ the subset of points with at least a coordinate in Σ. Let $\mathfrak{S}_m$ act on X^m by permutation of the factors. Then, for any subvariety $V \subset X^m$ of dimension d and such that $V \not\subset \Sigma'$, there exists an integer $q \geq 1$, and a $\mathfrak{S}_m$-invariant section $\sigma \in H^0(X^m, (\bigwedge^d \Omega_X)^{\otimes q} \otimes \mathcal{O}(-A^\sharp))^{\mathfrak{S}_m}$ such that $\sigma|_{\bigwedge^d T_{V^{\mathrm{reg}}}} \not\equiv 0$.*

Proof. Let us recall that $A^\sharp = \sum_{i=1}^{m} \mathrm{pr}_i^* A$. By Proposition 8.4, X^m satisfies the property $(H_{\Sigma', A^\sharp})$ so there exists $q_0 \geq 1$ and a section $\sigma_0 \in H^0(X^m, (\bigwedge^d \Omega_X)^{\otimes q_0})$, such that $\sigma_0|_{(\bigwedge^d T_{V^{\mathrm{reg}}})^{\otimes q_0}}$ vanishes at order r_0 along $A^\sharp|_V$.

Now, we let

$$\sigma = \bigotimes_{s \in \mathfrak{S}_m} s \cdot \sigma_0 \in H^0(X^m, (\bigwedge^d \Omega_X)^{\otimes m!\, q_0})$$

The section σ is $\mathfrak{S}_m$-invariant and vanishes along $A^\sharp$, hence satisfies our requirements. □

We now show the main hyperbolicity result of this section which implies Theorem 6 as an immediate corollary.

Theorem 20. *Let X be a complex projective manifold with $\dim X \geq 2$. Assume X satisfies $(H_{\Sigma, A})$ for some $\Sigma \subsetneq X$ and some ample divisor A on X.*

Then, any subvariety $V \subseteq X_m$ such that $\mathrm{codim} V \leq n-2$ and $V \not\subset X_m^{\mathrm{sing}} \cup \mathfrak{d}_1(\Sigma)$ is of general type.

Proof. Let $V \subset X_m$ be a d-dimensional variety satisfying the hypotheses above. We have then $d \geq (m-1)n + 2$. Let $X^m \xrightarrow{p} X_m$ be the canonical projection. We do not lose generality in replacing A by a high multiple (the condition $(H_{\Sigma,A})$ is preserved), and then moving it in its linear equivalence class, so we can assume that $V \not\subset |A|$.

By Proposition 8.5, for $q \gg 0$, there exists a section $\sigma \in \Gamma(X^m, (\bigwedge^p \Omega_{X^m})^{\otimes q})^{\mathfrak{S}_m}$, whose restriction to $(\bigwedge^d T_{p^{-1}(V^{\mathrm{reg}})})^{\otimes q}$ vanishes along the $\mathfrak{S}_m$-invariant ample divisor $A^\sharp$. This section descends to X_m; moreover, for any resolution of singularities $\widetilde{X}_m$, Lemma 8.1 shows that the Reid-Tai-Weissauer criterion of Proposition 2.1 is applicable. Hence, σ induces a section

$$\widetilde{\sigma} \in H^0(\widetilde{X}_m, (\bigwedge^d \Omega_{\widetilde{X}_m})^{\otimes q}).$$

Moreover, the restriction of $\widetilde{\sigma}$ to $\bigwedge^d T_{V^{\mathrm{reg}}}$ vanishes on the ample Cartier divisor $A_\flat$ defined so that $p^* A_\flat = A^\sharp|_V$.

Consider now a resolution of singularities $\widetilde{V} \xrightarrow{\varphi} V$. The pullback $\varphi^* \sigma$ induces a section of $K_{\widetilde{V}}$ that vanishes on the big divisor $\varphi^* A_\flat$. This implies that $K_{\widetilde{V}}$ is big, so V is of general type. □

Remark 8.6. The bound on $\dim V$ in Theorem 20 is sharp, as we can see from the following example. Let C be a genus 2 curve, and let Y be any $(n-1)$-dimensional variety with Ω_Y ample. Let $X = C \times Y$. This manifold satisfies property $(H_{\emptyset,A})$ for some ample divisor A by Propositions 8.3 and 8.4.

(1) In the case $m = 2$: let $f : S^2C \times Y \longrightarrow S^2(C \times Y) = S^2 X$ be the generically injective map

$$f([c_1, c_2], y_1, ..., y_{n-1}) = [(c_1, y_1, ..., y_{n-1}), (c_2, y_1, ..., y_{n-1})].$$

Since $g(C) = 2$, the variety $S^2(C)$ is birational to $\mathrm{Jac}(C)$ and thus $S^2 C \times Y$ is not of general type.

(2) In the case $m \geq 2$, consider the composition of $f \times \mathrm{Id}_{X^{m-2}} : S^2C \times Y \times X^{m-2} \longrightarrow S^2X \times X^{m-2}$ (where f is as above) and of the natural map $g : S^2X \times X^{m-2} \longrightarrow S^mX$.

We have $\dim S^2C \times Y \times X^{m-2} = n(m-1)+1$, and the image $V = (g \circ f)(S^2C \times Y \times X^{m-2})$ in X_m is not of general type, since $S^2C \times Y \times X^{m-2}$ is not.

Note that if the Green-Griffiths-Lang conjecture were true, then Theorem 20 would imply the following result.

Conjecture 8.7. *Let X be a complex projective manifold with Ω_X ample. Then,* $\operatorname{codim} \operatorname{Exc}(X_m) \geq n-1$.

We can use Theorem 20 to prove the following weaker statement, that gives geometric restrictions on the exceptional locus on non-hyperbolic *algebraic* curves in X_m. It gives also a more precise version of Corollary 1.6:

Corollary 8.8. *Assume that Ω_X is ample. Then, there exist countably many proper algebraic subsets $V_k \subsetneq X_m$ $(k \in \mathbb{N})$ containing the image of any non-hyperbolic algebraic curve. Moreover, the V_k can be chosen so that for any component W of $\mathfrak{D}_i(X_m)$ $(0 \leq i \leq n)$ containing V_k $(k \in \mathbb{N})$, we have* $\operatorname{codim}_W(V) \geq n-1$.

In particular (letting $i=0$ and $W = X_m$), we have $\operatorname{codim}_{X_m}(V_k) \geq n-1$ *for all $k \in \mathbb{N}$.*

Proof. As the irreducible components of each $\mathfrak{D}_i(X_m)$ identify to copies of X_{m-i}, it suffices to prove the last claim, and to show the result for curves C not included in $(X_m)_{\mathrm{sing}}$.

By [Kol95, Proposition 2.8], a Hilbert scheme argument shows that there exists:

(1) a locally topologically trivial family of normal varieties $p : \mathcal{V} \to B$, where B is a smooth scheme with *countably* many components;
(2) a morphism $f : \mathcal{V} \to X_m$,

such for any subvariety $V \subset X_m$, there exists $t \in B$ with $f(\mathcal{V}_t) = V$. Let $B_{\text{non hyp}} \subset B$ be the subset parametrizing curves of genus $g \leq 1$. Then, for any irreducible component V of $p^{-1}(B_{\text{non hyp}})$, the subvariety $\overline{f(V)} \subset X$ admits a dominant family of non-hyperbolic curves, and hence is not of general type. Since Ω_X is ample, Theorem 20 implies that $\operatorname{codim} \overline{f(V)} \geq n-1$ if $f(V) \not\subset (X_m)_{\mathrm{sing}}$. The property of $p : \mathcal{V} \to B$ finally implies that any non-hyperbolic curve $C \subset X_m$ with $C \not\subset (X_m)_{\mathrm{sing}}$ is included in one such $\overline{f(V)}$. This ends the proof. □

We can also prove Corollary 1.7, as the following statement, similar to [AA03, Corollary 4].

Corollary 8.9. *Assume that Ω_X is ample, and let $Y \subset X$ be a closed submanifold. Let $1 \leq l \leq d$ be integers. Assume that for a generic point $p \in Y_l \times X_{d-l}$, there exists a curve of geometric genus g in X such that all d coordinate points of p lie in C. Then if*

$$l \cdot \operatorname{codim} Y \leq \dim X - 2,$$

we have $g > d$.

Proof. Assume that $g \leq d$. By hypothesis, there exist $\mathcal{C} \to V$ a family of curves and a morphism $f : \mathcal{C} \longrightarrow X$, such that the image Z of $Y_l \times X_{d-l} \to X_d$ is dominated by the image of $S^d f : S^d\mathcal{C} \to X_d$. As in [AA03], we may replace V be a hyperplane section to assume that $S^d f$ is generically finite.

Since $g \leq d$, the family $\mathcal{S}^d\mathcal{C} \to V$ is a family of varieties which are not of general type (the fiber over t is a $\mathbb{P}^{d-g}$-bundle over $\mathrm{Jac}(C_t)$), and hence Z is not of general type as well. Since $\dim Z = \dim Y_l \times X_{d-l}$, Theorem 20 implies $\dim(Y_l \times X_{d-l}) < (d-1)\dim X + 2$, hence

$$\dim Y < \frac{1}{l}\left((l-1)\dim X + 2\right),$$

which gives the result. □

9. Metric methods

We will now present a metric point of view on these symmetric products of varieties, which will permit to give several applications to quotients of bounded symmetric domains.

We will use a metric hyperbolicity criterion similar to the one of [Cad18]. To express this criterion, we need first to introduce several constants bounding the Ricci curvature on subvarieties of the domain. Let us recall how to define these constants.

Let Ω be a bounded symmetric domain of dimension n, and let h_Ω be the Bergman metric on this domain. If $X, Y \in T_{\Omega,x}$ ($x \in \Omega$), we can define the *bi-sectional curvature* of h_Ω as

$$B(X,Y) = \frac{i\Theta(h_\Omega)(X, \overline{X}, Y, \overline{Y})}{||X||^2_{h_\Omega} ||Y||^2_{h_\Omega}}.$$

Fix $p \in \mathbb{N}$. Then, we define

$$C_p = - \max_{X \in T_{\Omega,x}} \max_{V \ni X, \dim V = p} \sum_{i=1}^{p} B(X, e_i), \tag{4}$$

where $V \subset T_{\Omega,x}$ runs among the p-dimensional subspaces containing X, and $(e_i)_{1\leq i \leq p}$ is any unitary basis of V. Since Ω is homogeneous, this constant does not depend on $x \in \Omega$.

Then, if we normalize the Bergman metric so that $C_n = 1$, we have a sequence of positive constants

$$0 < C_1 \leq C_2 \leq ... \leq C_n = 1.$$

These constants can be used to state the following criterion for the p-hyperbolicity of compactification of a quotient of Ω.

Proposition 9.1 (see [Cad18]). *Let M be a smooth projective manifold, and D, $E = \sum_i (1 - \alpha_i)E_i$ be $\mathbb{Q}$-divisors on X such that the support $|E| \cup |D|$ has normal crossings. Let $U = M - (|D| \cup |E|)$, and let h be a smooth Kähler metric on U, possibly degenerate. Let $p \in [\![1, \dim M]\!]$ and let $\alpha > \frac{1}{C_p}$ be a rational number. We make the following assumptions.*

- *(i) h is non-degenerate outside an algebraic subset $Z \subset M$, and is modeled after h_Ω on $U - Z$;*
- *(ii) the metric induced by h on $\bigwedge^d T_M$ has singularities near any point of $|E_i| - (|D| \cup Z)$ with coefficients of order at most $O(|z|^{2(\alpha_i - 1)})$;*

(iii) *there exists a non-zero section* s *of* $K_U^{\otimes l}$ *such that* $||s||^{2/l}_{(\det h^*)^l}$ *extends as a continuous function* u *on* M*, vanishing along* $E + D$ *at an order strictly larger than* $\frac{1}{C_p}$*. If* z *is a local equation for a component of weight* β *in* $D+E$*, this means that* $u = O(|z|^{\frac{\beta}{C_p}(1+\epsilon)})$ *for some* $\epsilon > 0$ *(recall that* $\beta = 1$ *for the components of* D*, and* $\beta = 1 - \alpha_i$ *for the* E_i*).*

Then,

(a) For any subvariety $V \subset M$ *with* $V \not\subset Z(s) \cup E \cup D \cup Z$ *and* $\dim V \geq p$, $\dim V$ *is of general type.*

(b) For any holomorphic map $f : \mathbb{C}^p \to M$ *with* $\mathrm{Jac}(f)$ *generically of maximal rank, we have* $f(\mathbb{C}^p) \subset Z(s) \cup E \cup D \cup Z$*.*

Proof. The metric h satisfies all the assumptions permitting to apply the proof of Theorem 2 and Theorem 8 of [Cad18]. Let us recall that the technique of this proof consists in forming the metric $\widetilde{h} = ||s||^{2\beta}_{(\det h^*)^m} h$ for an adequate $\beta > 0$. We then check that $\widetilde{h}$ induces a positively curved singular metric on the canonical bundle of a desingularization of any subvariety V as in the hypotheses. In the case of a map $f : \mathbb{C}^p \to M$, we apply the Ahlfors-Schwarz lemma (see [Dem12, 4.2]) to this metric to obtain a contradiction if $f(\mathbb{C}^p) \not\subset Z(s) \cup E \cup D \cup Z$. □

Remark 9.2. Assume that $X = {}_{\Gamma}\backslash^{\Omega}$ is a quotient by an arithmetic lattice, and let $q : M \to \overline{X}^{BB}$ be a log-resolution of the singularities of the Baily-Borel compactification of X. Let $U \subset X$ be the smooth locus, and E_i (resp. D_j) be the components of the exceptional divisor whose projection intersects X_{sing} (resp. whose projection lies in $\overline{X}^{BB} \setminus X$). For each i, let x_i be a generic point of the projection of E_i on $\overline{X}^{BB}$. Let $H_i \subset \Gamma$ be the isotropy group of x_i, and let α_i be such that the action of H_i on Ω satisfies the condition (I'_{x,d,α_i}) of Section 2.1. We associate the multiplicity α_i to E_i by putting $E = \sum_i (1 - \alpha_i) E_i$. We also let $D = \sum_i D_i$.

With this notation, as explained in [Cad18, Section 4], the hypotheses (i) and (ii) of Proposition 9.1 are satisfied. The condition (iii) is implied by the following more algebraic condition.

(iii') For $\alpha \in \mathbb{Q}_+^*$, let $L_\alpha = q^* K_{\overline{X}^{BB}} \otimes \mathcal{O}(-\alpha(D + E))$. Then L_α is effective for some $\alpha > \frac{1}{C_p}$.

Moreover, $Z(s)$ in (a) and (b) can then be replaced by the stable base locus $\mathbb{B}(L_\alpha)$.

Remark 9.3. We can generalize the conclusion *(b)* of Proposition 9.1 to the following situation. Assume that there exists a proper birational holomorphic map $q : M \to M_0$, where M_0 is a possibly singular complex variety. Then, under the assumption of the theorem, we can state the following:

(b') Let $W = q(Z(s) \cup E \cup D \cup Z)$. Then for any holomorphic map $f : \mathbb{C}^p \to M_0$ with $\mathrm{Jac}(f)$ generically of maximal rank, we have $f(\mathbb{C}^p) \subset W \cup (M_0)_{\text{sing}}$.

To prove this statement, assume by contradiction that there exists a $f : \mathbb{C}^p \to M_0$ that fails to satisfy the conclusion of *(b')*. Let C be a resolution of singularities of the main component of the fiber product $\mathbb{C}^p \times_{(f,q)} M$. Then, there exists a proper morphism $g : C \to \mathbb{C}^p$, birational outside a locally finite union of analytic subvarieties of $\mathbb{C}^p$, and there exists a natural map $h : C \to M$, generically non-degenerate, whose image intersects $U \setminus (Z \cup Z(s) \cup E)$. Construct $\widetilde{h}$ is as the proof of Proposition 9.1. Then, the metric $g^*\widetilde{h}$ on C is subject to the following version of the Ahlfors-Schwarz lemma.

Lemma 9.4. *Let $g : C \to \mathbb{C}^p$ be a proper holomorphic map, realizing an isomorphism outside a countable union of analytic subvarieties of $\mathbb{C}^p$. Then T_C cannot admit any singular metric h, with $\det h$ everywhere locally bounded, smooth on a dense open Zariski subset U, and satisfying the following inequality on U:*

$$dd^c \log \det h \geq \epsilon \omega_h \qquad (\epsilon > 0). \tag{5}$$

Proof. Assume by contradiction that there exists such a metric. We may assume that g is an isomorphism on some open subset $V \subset C$ containing U. We may then see h as a metric on $V \subset \mathbb{C}^p$, satisfying (5) on U. As $\det h$ is everywhere locally bounded on V, and since $dd^c \log \det h \geq 0$ on $U \subset V$, the function $\log \det h$ is psh on V. Besides, as $\mathbb{C}^p$ is normal, we have $\mathrm{codim}(\mathbb{C}^p \setminus V) \geq 2$, so $\log \det h$ extends *to the whole* $\mathbb{C}^p$ as a psh function, satisfying (5) in the sense of *currents*. This case is however ruled out by the standard Ahlfors-Schwarz lemma stated in [Dem12]. □

Our plan is to use the previous proposition in the case where X is a resolution of singularities of a symmetric product of a quotient of a bounded symmetric domain. To do so, we will need some estimates on the C_p when the domain is of the form Ω^m ($m \in \mathbb{N}$). The case $p = 1$ is fairly easy to settle: in this case, $-C_1$ is just the maximum of the holomorphic sectional curvature, and we have the following well-known result.

Proposition 9.5. *Let Ω be a bounded symmetric domain, and denote by $-\gamma$ the maximum of the holomorphic sectional curvature on Ω. Then we have*

$$C_1(\Omega^m) = \frac{1}{m} C_1(\Omega) = \frac{\gamma}{m}.$$

This can be checked directly by writing the formula for the bisectional curvature of Ω^m, or by remarking that by the polydisk theorem (see [Mok89]), it suffices to deal with the case where $\Omega = \Delta^n$. In this case the holomorphic sectional curvature is maximal in the direction of the long diagonals, and the formula can be easily derived.

We can use now use this result to study the case of ramified coverings of *smooth compact* quotients of bounded symmetric domains.

Proposition 9.6. *Let $Y = {}^{\Gamma}\backslash{}^{\Omega}$ be a smooth compact quotient, let $p : X \longrightarrow Y$ be a Block-Gieseker covering, and let $\delta = \frac{s}{r}$ be a positive rational number such that be such that $p^* K_Y^{\otimes r} = A^{\otimes s}$ for some very ample line bundle A. Let $W \subset X$ be the locus where p is non-étale.*

Then if $m \in \mathbb{N}$ is such that

$$\gamma\, \delta > 2m(m-1),$$

the variety X_m is Brody hyperbolic modulo $\mathfrak{d}_1(W)$.

Proof. Let $q : M \to X_m$ be a log-resolution of singularities, let $E \subset M$ be the exceptional locus, and Z be the preimage of $\mathfrak{d}_1(W)$. Let h_Y be the pullback of the Bergman metric on Y. This metric is smooth on Y, and non degenerate on $Y - W$. This metric induces in turn a natural metric on the smooth locus of Y_m, and by pullback, a smooth metric h on $M - E$.

Let us check that the conditions of Proposition 9.1 are satisfied for $p = 1$. Since h_Y is non-degenerate and modeled on h_Ω on $X - W$, the metric h is non-degenerate and modeled on h_{Ω^m} on $M - (E \cup Z)$, so the condition (i) is satisfied.

It follows directly from the discussion of Section 2.2 that the condition $(I'_{x,1,1})$ is satisfied for every $x \in X^m$. Hence, the condition (ii) holds for $E = \sum_i E_i$.

Let $x \in M - E$. By Proposition 7.3, for some $N \in \mathbb{N}$, there exists a section σ of $q^* A_\flat^{\otimes sN} \otimes (-\frac{Ns}{2(m-1)}|E|)$ that does not vanish at x. By hypothesis, the line bundles $(A_\flat)^{\otimes s}|_{X_m^{\mathrm{reg}}}$ and $K_{X_m^{\mathrm{reg}}}^{\otimes r}$ coincide. Thus, if N is divisible enough, σ can be seen as a section of the line bundle $(q^* K_{X_m} \otimes \mathcal{O}(-\frac{\delta}{2(m-1)} E))^{\otimes rN}$. Finally, the holomorphic sections of $q^* K_{X_m}^{\otimes rN}$ have bounded norm for the norm induced by h, which shows that (iii) is satisfied if $\delta > \frac{2(m-1)}{C_1(\Omega^m)} = \frac{2m(m-1)}{\gamma}$. This is precisely our hypothesis. Moreover, since $x \in M - E$ is arbitrary, the locus cut out by the sections σ is included in $M - E$. The conclusion follows as announced from Proposition 9.1. □

The following result of Hwang-To can be used to give a more explicit constant δ in the proposition above.

Theorem 21 ([HT00b]). *For any smooth compact quotient of a bounded domain X, there exists a finite étale cover X' such that $2K_{X'}$ is very ample.*

This gives immediately the following series of examples.

Example 4. Let $Y_0 = {}_\Gamma\backslash^{\Omega}$ be a smooth compact quotient, and let $Y_1 \longrightarrow Y_0$ be the étale cover provided by [HT00b]. Let $m \in \mathbb{N}^*$, and let q be an integer such that $q > 4\frac{m(m-1)}{\gamma}$.

Now let $X \xrightarrow{p} Y_1$ be a Bloch-Gieseker covering such that $p^*(K_{Y_1}^{\otimes 2}) = A^{\otimes q}$, with A very ample. Then, we have $\delta\gamma = \frac{q\gamma}{2} > 2m(m-1)$, so that X_m is Brody hyperbolic modulo $\mathfrak{d}_1(\mathrm{Sing}(p))$.

Example 5. For $1 \leq i \leq n$, let X_i be a smooth projective curve of genus $g \geq 2$, and fix some integer q. For all i, since $3K_{X_i}$ is very ample, we can perform a q-fold Bloch-Gieseker covering $p_i : X'_i \longrightarrow X_i$, so that $p_i^*(3K_{X_i}) = A_i^{\otimes q}$, with A_i very ample on X'_i.

Letting $X = X'_1 \times ... \times X'_n \xrightarrow{p} X_1 \times ... \times X_n = Y$, we have then $p^* K_Y^{\otimes 3} = A^{\otimes q}$, where $A = \bigotimes_{1 \leq j \leq n} p_j^* K_{X_j}$ is very ample on X. The manifold Y is a smooth compact quotient of Δ^n, and $\gamma = \frac{1}{n}$ for this domain. Proposition 9.6 shows then X_m is Brody hyperbolic modulo $(X_m)_{\mathrm{sing}}$ as soon as

$$q \geq 6m(m-1)n.$$

9.1. **Non-compact ball quotients.** In the case where the domain is the ball, it is possible to give explicit values for the constants C_p. The result can be stated as follows when $\dim \Omega \geq 5$.

Proposition 9.7. *We let $\Omega = \mathbb{B}^n$ for some $n \geq 5$. Let $m \in \mathbb{N}$, and fix $p \in [\![1, mn]\!]$. Let $k \in \mathbb{N}$ (resp. $d \in [\![0, n-1]\!]$) be the quotient (resp. the remainder) in the euclidean division of $p - 1$ by n. Then the value of $C_p(\Omega^m)$ is given by the table of Figure 1.*

<table>
<tr><td></td><td>$m-k=1$</td><td>$m-k=2$</td><td>$m-k=3$</td><td>$m-k=4$</td><td>$m-k\geq 5$</td></tr>
<tr><td>$d=0$</td><td rowspan="5">$\frac{d+2}{n+1}$</td><td colspan="4">$\frac{2}{(m-k)(n+1)}$</td></tr>
<tr><td>$d=1$</td><td>$\frac{23}{16}\frac{1}{n+1}$</td><td>$\frac{11}{12}\frac{1}{n+1}$</td><td>$\frac{21}{32}\frac{1}{n+1}$</td><td></td></tr>
<tr><td>$d=2$</td><td>$\frac{7}{4}\frac{1}{n+1}$</td><td colspan="3"></td></tr>
<tr><td>$d=3$</td><td>$\frac{31}{16}\frac{1}{n+1}$</td><td colspan="3">$\frac{2}{m-k-1}\frac{1}{n+1}$</td></tr>
<tr><td>$d\geq 4$</td><td colspan="4"></td></tr>
</table>

FIGURE 1. Values of C_p for the domain $(\mathbb{B}^n)^m$

Note the similarity with the case where Ω is the Siegel upper half-space (see [Cad18, Proposition 1.4]). We will prove Proposition 9.7 in Section 9.2. As an application, we can derive a proof of Theorem 7 as a corollary of our metric criterion:

Corollary 9.8. *Let $X = {}_{\Gamma}\backslash^{\mathbb{B}^n}$ be a ball quotient by a torsion free lattice with only unipotent parabolic elements, and let $\overline{X} = X \cup D$ be a smooth minimal compactification as constructed in [Mok12]. Let $m \geq 1$. Then :*

(a) Let $V \subset \overline{X}_m$ be a subvariety with $\operatorname{codim} V \leq n-6$ and $V \not\subset \mathfrak{d}_1(D) \cup (\overline{X}_m)_{\text{sing}}$. Then V is of general type.

(b) Let $p \geq n(m-1)+6$, and $f : \mathbb{C}^p \to \overline{X}_m$ be a holomorphic map such that $f(\mathbb{C}^p) \not\subset \mathfrak{d}_1(D) \cup (\overline{X}_m)_{\text{sing}}$. Then $\operatorname{Jac}(f)$ is identically degenerate.

Proof. Let $q : \widetilde{X} \to \overline{X}_m$ be a resolution of singularities. We may assume that $F = q^{-1}(\mathfrak{d}_1(D) \cup (\overline{X}_m)_{\text{sing}})$ is a simple normal crossing divisor. Let $\widetilde{D}$ denote the sum of components of F that project in $\mathfrak{d}_1(D)$, and E the sum of all other components.

Let $p \geq n(m-1)+6$ be an integer. By Proposition 9.7, since $p \geq n(m-1)+6$, the constant C_p is given by the first column of Figure 1, and $C_p = \frac{p-n(m-1)+1}{n+1} > \frac{2\pi}{n+1}$.

Let h be the metric induced on $U = \widetilde{X} \setminus (E+D)$. Let us check that the assumptions of Proposition 9.1 are satisfied, with $\Omega = (\mathbb{B}^n)^m$. (i) is obvious, taking $Z = \emptyset$. By Lemma 8.1, since $p \geq n(m-1)+2$, the condition $(I_{x,p})$ is satisfied above any singular point of $\overline{X}_m$, so Remark 9.2 implies that the hypothesis (ii) is satisfied with $\alpha_i = 1$ for any component $E_i \subset E$.

To prove (iii), we make use of [BT18], whose main result shows that the line bundle $K_{\overline{X}} + (1-\alpha)D$ is ample for any $\alpha > \frac{n+1}{2\pi}$. Let $\alpha \in]\frac{1}{C_p}, \frac{n+1}{2\pi}[$. Thus, for $l \in \mathbb{N}$ large enough, and any $x = (x_1, ..., x_m) \in \overline{X}^m \setminus \cup_{i=1} \mathrm{pr}_i^{-1}(D)$, we can find a section σ of $l\,(K_{\overline{X}} + (1-\alpha)D)$, such that $\sigma(x_i) \neq 0$ $(1 \leq i \leq m)$. Let $s^\sharp = \bigotimes_{1 \leq j \leq m} \mathrm{pr}_j^* \sigma$. This is a $\mathfrak{S}_m$-invariant section of $K_{X^m}^{\otimes l}$, which descends to a section s of $K_U^{\otimes l}$. Let $u = ||s||^{2/l}_{(\det h^*)^l}$.

We need to check the conditions on the growth of u near $E + \widetilde{D}$. First, u is bounded near any point of E since $||s^\sharp||_{(\det h_\Omega^*)^l}$ is continuous on the manifold X^m. Besides, by [Mum77, Theorem 3.1 and Proposition 3.4 (b)], the determinant of the Bergman metric on $K_{\overline{X}} + D$ has logarithmic growth near D. Hence, since σ, seen as a section of $l(K_{\overline{X}} + D)$, vanishes at order $l\alpha$ along D, then the function $||s^\sharp||^2_{\det h_\Omega^*} = \prod_i \mathrm{pr}_i^* ||s||_{h_{\mathbb{B}^n}}$ vanishes at any order $< l\alpha$ near $\mathrm{pr_i}^* D$. Now $||s^\sharp||^{2/l}_{(\det h_\Omega^*)^l} = u \circ \pi$, where $\pi : \overline{X}^m \to \overline{X}_m$ is the projection, so u vanishes at order α near any point of $\widetilde{D} \setminus E$. As $\alpha > \frac{1}{C_p}$, the section s satisfies the condition (iii).

Finally, since x was arbitrary outside $\bigcup_{1\leq i\leq m} \mathrm{pr}_i^* D$, we conclude from Proposition 9.1 that all p-dimensional varieties $V \subset \widetilde{X}$, not included in $E + \widetilde{D}$, are of general type. This proves (a).

The proof of (b) follows from the conclusion (b') in Remark 9.3, applied with $M = \widetilde{X}$, and $M_0 = \overline{X}_m$. □

9.2. **Computation of the curvature constants for the domain** $(\mathbb{B}^n)^m$**.** We now prove Proposition 9.7. We will proceed as in [Cad18], and introduce a certain combinatorial functional whose minimum will give us the value of $C_p(\Omega^m)$.

Definition 9.9. Let

$$\Delta_m = \{(r_1, ..., r_m) \in (\mathbb{R}_+)^m \mid \sum_{1\leq j\leq m} r_j = 1 \text{ and } r_1 \geq r_2 \geq ... \geq r_m\}.$$

Let $\underline{r} = (r_1, ..., r_m) \in \Delta_m$ and $\Gamma \subset [\![1, m]\!] \times [\![1, n]\!]$. Denote by k the number of elements of Γ in the first column. We assume that $k \leq m - 1$. We define:

$$\mathcal{F}(\underline{r}, \Gamma) = \begin{cases} 2 + \sum_{(i,j)\in\Gamma,\ i\geq 2} r_i & \text{if } k = m-1 \\ \\ 2\sum_{1\leq i\leq m} r_i^2 + 2\sum_{(i,1)\in\Gamma} r_i + \sum_{(i,j)\in\Gamma,\ j\geq 2} r_i & \text{if } k \leq m-2. \end{cases}$$

From now on, we fix a given minimizer $(\underline{r}, \Gamma)$ for $\mathcal{F}$, where $\underline{r} \in \Delta_m$, and Γ runs among cardinal $p-1$ subsets of $[\![1, m]\!] \times [\![1, n]\!]$ with less than $m-1$ elements on the first column. Let k be the number of these elements. We will assume that $(\underline{r}, \Gamma)$ is chosen among all the minimizers so that

(1) $\underline{r} = (r_1, ..., r_m)$ has the maximal number of zero components ;
(2) among all minimizing couples $(\underline{r}, \Gamma)$ satisfying (1), Γ is chosen so that k *is maximal.*

We can make a simple remark on the geometry of Γ. Let

$$\Pi = \Gamma \cap ([\![1, m]\!] \times [\![2, n]\!])$$

be the set of elements of Γ which are outside of the first column. For each $i \in [\![1, m]\!]$, denote by b_i the number of elements of Π which are on the i-th line. Then, since $r_1 \geq ... \geq r_m$, we see from the formula for $\mathcal{F}$ that we may suppose that the elements of Π are the largest possible in the lexicographic order. This implies that for some $q \in [\![0, m]\!]$, $d \in [\![0, n-2]\!]$, we have $b_{m-j} = n - 1$ $(0 \leq j \leq q-1)$, $b_{m-q} = d$, and $b_{m-j} = 0$ $(m \leq j \leq q+1)$.

Lemma 9.10. *Let l be the maximal integer such that $r_{m-l+1} = ... = r_m = 0$. We have $l = k$.*

Proof. The proof is exactly the same as the one of [Cad18, Lemma 3.8], replacing g by m, Γ_0 by Γ, and "off-diagonal" by "off the first column". □

The previous proof relies on the following lemma, which will be used frequently in the following.

Lemma 9.11 (see [Cad18, Lemma 3.9])**.** *Let $a_1 \leq ... \leq a_m$ be non-negative integers, and let t be the smallest integer such that $\sum_{i=1}^{t}(a_t - a_i) \geq 4$ (let $t = m+1$ if there is no such integer). Let $\underline{r} \in \Delta_m$ be a minimizer for the quadratic form*

$$Q(r_1, ..., r_m) = 2\sum_{i=1}^{m} r_i^2 + \sum_{i=1}^{m} a_i r_i.$$

Then $r_t = ... = r_m = 0$.

We will now compute the several possible values for the minimum $\mathcal{F}(\underline{r}, \Gamma)$. We will proceed by distinguishing along the value of k. There is one simple first case.

Lemma 9.12. *If $k = m - 1$, then*

$$\mathcal{F}(\underline{r}, \Gamma) = 2 + b_1.$$

Proof. In this case, we have

$$\mathcal{F}(\underline{r}, \Gamma) = 2 + \sum_{1 \leq i \leq m} b_i r_i.$$

Recall that the b_i are non-decreasing. Since $\underline{r}$ must be an extremum of the function $\mathcal{F}(\cdot, \Gamma)$, we see that we may chose $\underline{r} = (1, 0, ..., 0)$, which gives the result. □

We will now assume that $k \leq m - 2$, and distinguish several subcases.

Case 0. $q < k$.

In this situation, since $r_{m-k+1} = ... = r_m = 0$, we simply have $\mathcal{F}(\underline{r}, \Gamma) = 2\sum_{i=1}^{m-k} r_i^2$. The minimum is then reached for $(r_1, ..., r_m) = (\frac{1}{m-k}, ..., \frac{1}{m-k}, 0, ..., 0)$, and the value of the minimum is

$$\mathcal{F}(\underline{r}, \Gamma) = \frac{2}{m-k}.$$

Assumption. In the remaining cases 1 and 2 below, we will assume that $q \geq k$, which means that $r_{m-q} \neq 0$.

Case 1. $d \geq 1$.

By our previous description of the shape of Π, this implies that two subcases are *a priori* possible.

Case 1a. $q \geq k + 1$, i.e. the line $\{m - k\} \times [\![2, n-1]\!]$ is included in Γ.

Case 1b. $q = k$ i.e. the only elements of $[\![1, m-k]\!] \times [\![2, n-1]\!]$ in Γ are the d last elements of $\{m - k\} \times [\![2, n-1]\!]$.

Lemma 9.13. *The case 1a. cannot occur.*

Proof. In the case 1a, since $r_{m-k} \neq 0$, Lemma 9.11 shows that $\sum_{i \leq m-k}(b_{m-k} - b_i) \leq 3$. Hence, all elements of $[\![1, m-k]\!] \times [\![2, n-1]\!]$ are in Γ, except δ elements on the first line, with $1 \leq \delta \leq 3$. (If $\delta = 0$, we would have $d = 0$).

This shows that $b_1 = n - 1 - \delta$, with $1 \leq \delta \leq 3$, and $b_j = n - 1$ $(2 \leq j \leq m - k)$. In this setting, the minimizer $\underline{r}$ is of the form $(x, y, ..., y, 0, ..., 0)$ where y is repeated $m - k - 1$ times, and $x + (m - k - 1)y = 1$. Let $b = m - k - 1$.

The minimum then equals

$$\mathcal{F}(\underline{r}, \Gamma) = 2x^2 + 2by^2 + (n-1) - \delta x.$$

We claim that $b \leq 2$. Indeed, if $b \geq 3$, since $n-1 \geq 4$, we can remove $4-\delta$ elements on the first line of Γ, to get a new set Γ'. If $\underline{r}' \in \Delta_m$ is a minimizer for the functional $\mathcal{F}(\cdot, \Gamma')$, we have $r'_2 = ... = r'_m = 0$ by Lemma 9.11. Since $b \geq 3$, there is enough room on the first column of Γ' to add back the $4-\delta$ elements, which gives a new set Γ'' with strictly more elements on the first column than Γ. Now

$$\mathcal{F}(\underline{r}', \Gamma'') = \mathcal{F}(\underline{r}', \Gamma') \leq \mathcal{F}(\underline{r}, \Gamma') \leq \mathcal{F}(\underline{r}, \Gamma).$$

(The first equality comes from the fact the $r'_2 = ... = r'_m = 0$, and the inequalities are obvious since all r_i are non-negative). This gives a contradiction with our choice of $(\underline{r}, \Gamma)$.

The same computation as in [Cad18, Lemma 3.14] shows that the case $b = 1$ is impossible.

Let us finally exclude the case $b = 2$. In this situation $\underline{r} = (x, y, y, 0, ..., 0)$ minimizes $\mathcal{F}(\underline{r}, \Gamma) = 2x^2 + 4y^2 + (n-1) - \delta x$, with the constraint $x + 2y = 1$. We check that the minimum is equal to

$$n - \frac{(2+\delta)^2}{12}.$$

Since $b = 2$, there are two elements of $[\![1, m]\!] \times \{1\}$ which are not in Γ, and we can move two elements of the first row Γ to get a new set Γ' with $m-1$ elements in the first column. Letting $\underline{r}' = (1, 0, ..., 0)$, we have

$$\begin{aligned} \mathcal{F}(\underline{r}', \Gamma') &= 2 + (n-1) - (\delta + 2) \\ &= n - 1 - \delta \\ &< n - \frac{(2+\delta)^2}{12} = \mathcal{F}(\underline{r}, \Gamma), \end{aligned}$$

since $\delta \in \{1, 2, 3\}$. This is a contradiction. □

Lemma 9.14. *In the case 1b, there are only 5 possibilities, which are given in the table of Figure 2.*

Proof. In this case, we have $b_{m-q} = d$, and this is the only non-zero b_j with $j \leq m - l$. By Lemma 9.11 again, we have $d(m-k-1) \leq 3$ since $r_{m-k} \neq 0$. Since $d \neq 0$ and $m - k \geq 2$ in the case under study, this gives only only five possibilities. The corresponding values for the minimum of $\mathcal{F}(\underline{r}, \Gamma) = 2\sum_{j=1}^{m-k} r_j^2 + d r_{m-k}$ were computed in [Cad18, Case 2]. □

	$m-k=2$	$m-k=3$	$m-k=4$
$d=1$	$\frac{23}{16}$	$\frac{11}{12}$	$\frac{21}{32}$
$d=2$	$\frac{7}{4}$		
$d=3$	$\frac{31}{16}$		

FIGURE 2. Possible values of the minimum of $\mathcal{F}$ in the case 1b

There is only one remaining case.

Case 2. $d = 0$.

Lemma 9.15. *Case 2 cannot occur unless* Γ *is of the form* $[\![m-k+1,m]\!]\times[\![1,n]\!]$. *The value of the minimum is then*

$$\mathcal{F}(\underline{r},\Gamma)=\frac{2}{m-k}.$$

Proof. If Γ is not of the prescribed form, we have

$$\mathcal{F}(\underline{r},\Gamma)=2\sum_{1\leq j\leq m-k} r_j^2+(n-1)\sum_{j=m-q+1}^{m-k} r_j,$$

with $q<k$. Applying another time Lemma 9.11, since $r_{m-k}\neq 0$, we have $(n-1)(m-q)\leq 3$ for all $t\geq 1$. As we assumed that $n\geq 5$, this implies that $q=m$, i.e. Γ contains all the elements which are not on the first column. The minimum is then reached for $\underline{r}$ of the form $\underline{r}=(\frac{1}{m-k},...,\frac{1}{m-k},0,...,0)$ $(1/(m-k)$ repeated $m-k$ times), and its value is

$$\mathcal{F}(\underline{r},\Gamma)=\frac{2}{m-k}+(n-1).$$

However, this is absurd. Indeed, let Γ' be obtained from Γ by moving elements to its $m-k-1$ empty slots on the first column (recall that we consider sets with at most $m-1$ elements on the first column).

If $m-k\geq 3$, we may then assume that Γ' has less than $(n-1)-2$ elements on the first line. Letting $\underline{r}'=(1,0,...,0)$, we get

$$\mathcal{F}(\underline{r}',\Gamma')\leq 2+(n-3)<\frac{2}{m-k}+(n-1)=\mathcal{F}(\underline{r},\Gamma),$$

which is a contradiction.

If $m-k=2$, we may move one element, and assume that Γ' has $n-2$ elements on the first line. Then, letting again $\underline{r}'=(1,0,...,0)$, we get

$$\mathcal{F}(\underline{r}',\Gamma')=2+(n-2)=\frac{2}{m-k}+(n-1)=\mathcal{F}(\underline{r},\Gamma).$$

This is again a contradiction, since we assumed that Γ had the maximal number of elements on the first column. ☐

Putting everything together, we have proved the following.

Proposition 9.16. *Let* $p\in[\![1,mn]\!]$. *Let* $k=\lfloor\frac{p-1}{n}\rfloor$, *and* $d=p-1-kn$. *Let* $(\underline{r},\Gamma)$ *be a minimizer for* $\mathcal{F}$, *where* $\underline{r}\in\Delta_m$, *and* $\Gamma\subset[\![1,m]\!]\times[\![1,n]\!]$ *is a cardinal* $p-1$ *subset with less that* $m-1$ *elements on the first column. Then*

1. *the value of* $\mathcal{F}(\underline{r},\Gamma)$ *is given by the table of Figure 3 ;*
2. *we may choose* $(\underline{r},\Gamma)$ *so that the elements of* Γ *in the first column are the* $(j,1)$ *with* $j\geq m-k+1$, *and so that* $r_{m-k+1}=...=r_m=0$.

We will now show that the previously computed maxima permit to give the constant C_p. Let us recall how this constant can be computed.

In the following, if Ω is a bounded symmetric domain, and X is a vector tangent to Ω, we will denote by $B_0^{\Omega}(X,\cdot)$ the following bilinear form:

$$B_0^{\Omega}(X,\cdot):Y\longmapsto i\Theta(h_\Omega)(X,\overline{X},Y,\overline{Y}).$$

Let $X\in T_{\Omega,0}$ be a unitary vector. Let $V\subset T_{\Omega^m,0}$ be a d-dimensional vector space containing X. We now assume that the pair (X,V) realizes the maximum of (4).

<table>
<tr><th></th><th>$m-k=1$</th><th>$m-k=2$</th><th>$m-k=3$</th><th>$m-k=4$</th><th>$m-k\geq 5$</th></tr>
<tr><td>$r=0$</td><td rowspan="5">$d+2$</td><td colspan="4">$\frac{2}{m-k}$</td></tr>
<tr><td>$d=1$</td><td>$\frac{23}{16}$</td><td>$\frac{11}{12}$</td><td>$\frac{21}{32}$</td><td></td></tr>
<tr><td>$d=2$</td><td>$\frac{7}{4}$</td><td colspan="3" rowspan="2">$\frac{2}{m-k-1}$</td></tr>
<tr><td>$r=3$</td><td>$\frac{31}{16}$</td></tr>
<tr><td>$d\geq 4$</td><td colspan="4"></td></tr>
</table>

FIGURE 3. Values of the maxima of $\mathcal{F}$

We let $\mathrm{Aut}(\mathbb{B}^n)^m$ act on Ω so that X decomposes in the direct sum $T_{\Omega,0} = (T_{\mathbb{B}^n,0})^{\oplus m}$ as $X = (\alpha_1 e_1^1, ..., \alpha_m e_1^m)$, where $(e_1^i, ..., e_n^i)$ denotes a unitary basis of the i-th factor $T_{\mathbb{B}^n}$. We let $r_i = \alpha_i^2$ $(1 \leq i \leq m)$, so that $\sum_{1\leq i\leq m} r_i = 1$. We may assume that $r_1 \geq r_2 \geq ... \geq r_m$.

By our choice of (X, V), we have

$$C_p = -B_0(X, X) + \sum_{\lambda \in S(V)} \lambda, \tag{6}$$

where $S(V)$ is the set of the $p-1$ eigenvalues of the restriction of the hermitian form $-B_0(X, \cdot)$ to $X^\perp \cap V$ (with multiplicities). We let $W \subset V$ be a $(p-1)$-dimensional vector subspace, spanned by corresponding eigenvectors, so that $V = \mathbb{C}X \overset{\perp}{\oplus} W$.

Let us now explain how to compute the eigenvalues of the hermitian form $B_0^\Omega(X, \cdot)$ on the space $T_{\Omega,0}$. First, it is easy to show that for $U = (U_1, ..., U_m)$, $V = (V_1, ..., V_m)$ in $T_{\Omega,0}$, we have

$$B_0^\Omega(U, V) = \sum_{1\leq m} B_0^{\mathbb{B}^n}(U_i, V_i).$$

To simplify the computation, we will temporarily adopt a new normalization on $h_{\mathbb{B}^n}$, so that for any $U \in T_{\mathbb{B}^n,0}$, the eigenspaces of $-B_0^{\mathbb{B}^n}(U, \cdot)$ are

$$\begin{cases} \mathbb{C} \cdot U & \text{for the eigenvalue } 2||U||^2; \\ U^\perp \subset T_{\mathbb{B}^n} & \text{for the eigenvalue } ||U||^2. \end{cases}$$

Thus, with this normalization, the eigenvalues of $B_0^\Omega(X, \cdot)$ are $2r_i$ (with multiplicity 1, and eigenvector e_1^i) and r_i (with multiplicity $n-1$, with eigenvectors $e_2^i, ..., e_n^i$), for $1 \leq i \leq m$.

Proposition 9.17. *With the above normalization, the constant C_p is equal to the minimum of $\mathcal{F}$.*

The proof is the same as in [Cad18], so we will only sketch it briefly.

Lemma 9.18. *We have $C_p \geq \min_{\underline{r},\Gamma} \mathcal{F}(\underline{r}, \Gamma)$, where $\underline{r} \in \Delta_m$, and $\Gamma \subset [\![1, m]\!] \times [\![1, n]\!]$ runs among the cardinal $p-1$ subsets with less that $m-1$ elements on the first column.*

Proof. We can decompose $W = W_1 \overset{\perp}{\oplus} W_2$, where

$$W_1 \subset \overset{\perp}{\bigoplus_{1\leq i\leq m}} \mathbb{C}e_1^i, \text{ and } W_2 \subset \overset{\perp}{\bigoplus_{1\leq i\leq m}} \mathrm{Vect}(e_2^i, ..., e_n^i).$$

Let $k = \dim W_1$. By the description above of the eigenvalues of $B_0^\Omega(X, \cdot)$, we see that W_2 is spanned by $p - 1 - k$ eigenvectors corresponding to the eigenvalues r_i $(1 \leq i \leq m)$.

Let S_1 be the sum of the k smallest of the $2r_i$, and S_2 be the sum of the k-th smallest of the eigenvalues of $-B_0(X, \cdot)$ on W_2. Then

$$\begin{aligned} C_p &= -B_0(X, X) - \mathrm{Tr}B_0(X, \cdot)|_{W_1} - \mathrm{Tr}B_0(X, \cdot)|_{W_2} \\ &\geq -B_0(X, X) + S_1 + S_2 = 2\sum_{i\geq i} r_i^2 + S_1 + S_2. \end{aligned}$$

The eigenvalues appearing in S_1 and S_2 can be indexed by a subset $\Gamma \subset [\![1, m]\!] \times [\![1, n]\!]$, with k-elements of the first column corresponding to the k-th smallest $2r_i$, and the elements (i, j) to the r_j if $j \geq 2$.

There are two cases to distinguish. First, if $k \leq m - 1$, what has just been said shows that $C_p \geq \mathcal{F}(\underline{r}, \Gamma)$.

Now, if $k = m - 1$, then $\mathbb{C}X \overset{\perp}{\oplus} W_1 = \bigoplus_{i=1}^m \mathbb{C} \cdot e_1^i$, so

$$\begin{aligned} -B_0(X, X) - \mathrm{Tr}B_0(X, \cdot)|_{W_1} &= \mathrm{Tr}\left(-B_0(X, \cdot)|_{\bigoplus_{i=1}^m \mathbb{C}\cdot e_1^i}\right) \\ &= 2. \end{aligned}$$

C_p is equal to the first case of the definition of $\mathcal{F}$ in Definition 9.9, so $C_p = \mathcal{F}(\underline{r}, \Gamma)$. □

Lemma 9.19. *We have* $\min_{\underline{r},\Gamma} F(\underline{r}, \Gamma) \geq C_p$.

Proof. Let $\underline{r}$ and Γ realizing this minimum. Let W be the $p - 1$-dimensional space spanned by the eigenvectors corresponding to the elements of Γ, and let $X = (\sqrt{r_1}e_1^1, ..., \sqrt{r_m}e_1^m)$. By Proposition 9.16 (2), we see that $W \subset X^\perp$, so if we let $V = \mathbb{C} \oplus W$, we have

$$\begin{aligned} -\mathrm{Tr}B_0(X, \cdot)|_V &= -B_0(X, X) - \mathrm{Tr}\, B_0(X, \cdot)|_W \\ &= \mathcal{F}(\underline{r}, \Gamma). \end{aligned}$$

As C_p is defined to be the minimum of the left hand side for all X and V with $\dim V = p$ and $X \in V$ unitary, this shows that $\mathcal{F}(\underline{r}, \Gamma) \geq C_p$. □

Thus, Figure 3 gives the constants C_p with our simplifying normalization. To obtain the table 1, for which the normalization is chosen so that $C_{nm} = 1$, we must replace C_p by $\frac{C_p}{C_{nm}}$. In our current normalization, we have $C_{nm} = n + 1$ according the the first column of Table 3. This ends the proof of Proposition 9.7.

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