FOLIATIONS, SHIMURA VARIETIES AND THE GREEN-GRIFFITHS-LANG CONJECTURE

Abstract. Foliations have been recently a crucial tool in the study of the degeneracy of entire curves on projective varieties of general type. In this note, considering the Green-Griffiths locus, we explain how to deal with the case where there is no natural foliation to start with. As an application, we show that for most quotients of classical bounded symmetric domains, the Green-Griffiths locus is the whole variety.

1. Introduction

Foliations are known to play an important role in the study of subvarieties of projective varieties. One beautiful example is the proof of the following theorem of Bogomolov.

Theorem 1.1 ([1]). *Let X be a projective surface of general type such that $c_1^2 > c_2$. Then X has only finitely many rational or elliptic curves.*

The numerical positivity $c_1^2 > c_2$ gives the existence of global symmetric tensors on X and reduces the problem to the study of rational or elliptic algebraic leaves of foliations on a surface of general type. This result fits into the general study of hyperbolic (in the sense of Kobayashi) properties of algebraic varieties as illustrated by the famous Green-Griffiths-Lang conjecture

Conjecture 1.2. *Let X be a projective variety of general type. Then there exists a proper Zariski closed subset $Y \subsetneq X$ such that for all non-constant holomorphic curve $f : \mathbb{C} \to X$, we have $f(\mathbb{C}) \subset Y$.*

Although this is still largely open, even for surfaces, McQuillan has extended Bogomolov's theorem to transcendental leaves proving Green-Griffiths-Lang conjecture for surfaces of general type with positive second Segre number $c_1^2 - c_2$ [6].

Due to ideas which can be traced back to Bloch, it is now classical that algebraic differential equations can be used to attack such problems as follows. Holomorphic maps $f : U \subset \mathbb{C} \to X$ canonically lift to projectivized jets spaces $f_{[k]} : U \to P(J_kX)$. Let A be an ample line bundle on X, $p_k : P(J_kX) \to X$ the natural projection and $B_{k,l} \subset P(J_kX)$ be the base locus of the line bundle $\mathscr{O}_{P(J_kX)}(l) \otimes p_k^*A^{-1}$. We set $B_k = \bigcap_{l\in\mathbb{N}} B_{k,l}$ and $GG = \bigcap_{k\in N} p_k(B_k)$, the Green-Griffiths locus. The strategy is based on the fundamental vanishing theorem

Theorem 1.3 (Green-Griffiths, Demailly, Siu-Yeung). *Let $f : \mathbb{C} \to X$ be a non constant holomorphic curve. Then $f_{[k]}(\mathbb{C}) \subset B_k$ for all k. In particular, we have $f(\mathbb{C}) \subset GG$.*

On the positive side of this strategy, there is the recent result by Demailly which for our purposes can be stated as follows.

Theorem 1.4 ([2]). *Let X be a projective variety of general type. Then for some $k \ggg 1$, $B_k \neq P(J_k X)$.*

In other words, all non-constant holomorphic curves $f : \mathbb{C} \to X$ in a projective variety of general type satisfy a non-trivial differential equation $P(f, f', \dots, f^k) \equiv 0$.

On the less optimistic side, it is recently shown in [3] that one cannot hope $GG \neq X$ in general.

From [3] Theorem 1.3, one can extract the following criterion

Theorem 1.5. *Let X be a projective variety endowed with a holomorphic foliation by curves $\mathscr{F}$. If the canonical bundle of the foliation $K_{\mathscr{F}}$ is not big then $GG = X$.*

This produces examples of (hyperbolic!) projective varieties of general type whose Green-Griffiths locus satisfies $GG = X$ (see [3] for details).

Example 1.6. *Let $X = C_1 \times C_2$ a product of compact Riemann surfaces with genus $g(C_i) \geq 2$. Then X is of general type, hyperbolic and $GG = X$.*

Example 1.7. *Let $X = \Delta \times \Delta / \Gamma$ be a smooth compact irreducible surface uniformized by the bi-disc. X is naturally equipped with 2 non-singular foliations $\mathscr{F}$ and $\mathscr{G}$ with Kodaira dimension $-\infty$. Then X is of general type, hyperbolic and $GG = X$.*

Example 1.8. *Let $X = \Delta^n / \Gamma$ be a (not necessarily compact) quotient of the polydisc by an arithmetic lattice commensurable with the Hilbert modular group. Then $\overline{X}$ any compactification satisfies $GG = \overline{X}$.*

In all previous examples we have natural foliations on the manifold that one uses in a crucial way to obtain that the Green-Griffths locus is the whole manifold. It is therefore tempting to think that the absence of these natural foliations could be an obstruction to the Green-Griffiths locus covering the whole manifold. In fact in [3], using analytic techniques inspired by [7], we show that projective manifolds uniformized by bounded symmetric domains of rank at least 2 satisfy $GG = X$.

The goal of this note is to study the example of quotients of irreducible classical bounded symmetric domains, also in the non-compact case, using the arithmetic data of the Shimura varieties associated to these quotients.

2. Arithmetic lattices in classical groups

Let D be an irreducible symmetric bounded domain in $\mathbb{C}^N$ and Γ a torsion-free lattice in the simple real Lie group $G = \mathrm{Aut}^0(D)$. Let $\overline{X}$ be a compactification of $X = D/\Gamma$. Let us recall the description of the irreducible classical bounded symmetric domains $D = G/K$:

- $D^I_{p,q} = \{Z \in M(p, q, \mathbb{C}) / I_q - Z^* Z > 0\}$,
- $D^{II}_n = \{Z \in D^I_{n,n} / Z^t = -Z\}$,

- $D_n^{III} = \{Z \in D_{n,n}^I / Z^t = Z\}$,
- $D_n^{IV} = \{Z \in SL(2n, \mathbb{C}) / Z^t Z = I_{2n}, Z^* J_n Z = J_n\}$.

The group G can be described as follows:

- $D_{p,q}^I$: G is the special unitary group of a hermitian form on $\mathbb{C}^{p+q}$ of signature (p,q), $q \leq p$.
- D_n^{II}: G is the special unitary group of a skew-hermitian form on $\mathbb{H}^n$.
- D_n^{III}: G is the special unitary group of a skew-symmetric form on $\mathbb{R}^{2n}$.
- D_n^{IV}: G is the special unitary group of a symmetric bilinear form on $\mathbb{R}^{n+2}$ of signature $(n,2)$.

Each of the $\pm$ symmetric/hermitian forms is isotropic, and if $t = \mathrm{rank}_{\mathbb{R}}(G)$, the maximal dimension of a totally isotropic subspace is $t = q, [\frac{n}{2}], n, 2$ in the cases I, II, III, and IV respectively.

A deep theorem of Margulis states that if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then any discrete subgroup $\Gamma \subset G$ is arithmetic. That means that there is an algebraic $\mathbb{Q}$-group $\mathbb{G}$ with $\Gamma \subset \mathbb{G}_{\mathbb{Q}} \subset \mathbb{G}_{\mathbb{R}} = G$ and a rational representation $\rho : \mathbb{G}_{\mathbb{Q}} \to GL(V_{\mathbb{Q}})$ such that $\rho^{-1}(GL(V_{\mathbb{Z}}))$ and Γ are commensurable.

So, from now on, we suppose that we are in the situation where $\mathbb{G}$ is a $\mathbb{Q}$-group such that $\mathbb{G}_{\mathbb{R}} = G$ is one of the above classical real Lie group and of real rank at least 2.

We shall prove that in this situation the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$ (except in a few cases that we cannot deal with yet described below).

Let us explain the idea to prove such a result. Recall that in the case of the polydisc $D = \Delta^r$, we used in an essential way the existence of natural holomorphic foliations on the manifold coming from factors of the polydisc.

So, to prove such a result for quotients of irreducible domains (e.g. Siegel modular varieties), we need to find something which replaces the existence of natural foliations. The idea is to use, instead of foliations on D, the existence of many embedded polydiscs. This is motivated by the polydisc theorem ([7] ch.5, thm.1) which tells that there exists a totally geodesic submanifold E of D such that $(E, g_{|D})$ is isometric to a Poincaré polydisc (Δ^t, g_{Δ^t}), and $D = K.E$ where K is a maximal compact subgroup of G and g is the Bergman metric. If $M \subset G$ is the hermitian subgroup associated to E, to deduce that the restriction $\Gamma \cap M$ induces the existence of an algebraic subvariety we need some arithmetic conditions. In particular, it is sufficient that M comes from a reductive $\mathbb{Q}$-subgroup $\mathbb{M} \subset \mathbb{G}$ (see for example [5]). So, one reduces the problem to the existence of $\mathbb{Q}$-subgroups inducing a dense subset of subvarieties whose universal cover are polydiscs.

Let us recall the classification of classical $\mathbb{Q}$-groups of hermitian type. They are obtained by restriction of scalars $\mathbb{G} = Res_{k|\mathbb{Q}}\mathbb{G}'$ for an absolutely simple $\mathbb{G}'$ over k a totally real number field. The classification is the following (see [8] and [9]):

(1) Unitary type:

U.1) $\mathbb{G}' = SU(V, h)$, where V is an n-dimensional K-vector space, $K|k$ an imaginary quadratic extension, and h is a hermitian form. Then $G(\mathbb{R}) \cong \prod SU(p_\nu, q_\nu)$, where (p_ν, q_ν) are the signatures at infinite places.

U.2) $\mathbb{G}' = SU(V, h)$ where D is a division algebra of degree $d \geq 2$, central simple over K with a $K|k$-involution and V is an n-dimensional right D-vector space with hermitian form h. Then $\mathbb{G}(\mathbb{R}) \cong \prod SU(p_\nu, q_\nu)$.

(2) Orthogonal type:

O.1) $\mathbb{G}' = SO(V, h)$, V a k-vector space of dimension $n+2$, h a symmetric bilinear form such that at an infinite place ν, h_ν has signature $(n, 2)$ and $\mathbb{G}'_\nu(\mathbb{R}) \cong SO(n, 2)$.

O.2) $\mathbb{G}' = SU(V, h)$, V a right D-vector space of dimension n, h is a skew-hermitian form, D is a quaternion division algebra, central simple over k, and at an infinite place ν, either

(i) $D_\nu \cong \mathbb{H}$ and $\mathbb{G}'_\nu(\mathbb{R}) \cong SO(n, \mathbb{H})$, or

(ii) $D_\nu \cong M_2(\mathbb{R})$ and $\mathbb{G}'_\nu \cong SO(2n-2, 2)$.

(3) Symplectic type:

S.1) $\mathbb{G}' = Sp(2n, k)$ and $G'(\mathbb{R}) \cong Sp(2n, \mathbb{R})$

S.2) $\mathbb{G}' = SU(V, h)$, where V is an n-dimensional right vector space over a totally indefinite quaternion division algebra, and h is a hermitian form on V. Then $\mathbb{G}'(\mathbb{R}) \cong Sp(2n, \mathbb{R})$.

We shall prove the following result.

Theorem 2.1. *Let $X = D/\Gamma$ be an irreducible arithmetic quotient of a bounded symmetric domain of real rank at least 2. If X is of type U.1, O.1 ($n \geq 4$), O.2)(i), S.1, or S.2 then the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$.*

In other words, the only cases remaining are $U.2$ and $O.2)(ii)$.

3. The case of Siegel modular varieties

An interesting case is the case of Siegel modular varieties corresponding to the symplectic case $S.1$ in the above classification. The proof of the following result will be very explicit giving the ideas of the general result.

Theorem 3.1. *Let $X = D/\Gamma$, where $D = D_n^{III}$ and $\Gamma \subset Sp(2n, \mathbb{R})$ commensurable with $Sp(2n, \mathbb{Z})$, $n \geq 2$, then the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$.*

Proof. There is a totally geodesic polydisk $\Delta^n \hookrightarrow D$,

$$z = (z_1, \dots, z_n) \to z^* = diag(z_1, \dots, z_n)$$

of dimension n consisting of diagonal matrices $\{Z = (z_{ij})/z_{ij} = 0 \text{ for } i \neq j\} \subset D$. This corresponds to an embedding $Sl(2, \mathbb{R})^n \hookrightarrow Sp(2n, \mathbb{R})$: $M = (M_1, \dots, M_n) \to M^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$, where $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, and $a^* = diag(a_1, \dots, a_n)$ is the corresponding diagonal matrix.

More generally, taking $A \in Gl(n, \mathbb{R})$ one can consider the map $\Delta^n \hookrightarrow D_n^{III}$, given by

$$z = (z_1, \dots, z_n) \to A^t z^* A.$$

In order to take quotients, one defines

$$\Gamma_A := \{M \in Sl(2, \mathbb{R})^n / \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix} M^* \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1} \in \Gamma\}.$$

Indeed we have a modular embedding

$$\varphi_A : \Delta^n / \Gamma_A \to X.$$

Considering a totally real number field $K/\mathbb{Q}$ of degree n with the embedding $K \hookrightarrow \mathbb{R}^n$, $\omega \to (\omega^{(1)}, \dots, \omega^{(n)})$, the matrices $A = (\omega_i^{(j)})$ where $\omega_1, \dots, \omega_n$ is a basis of K have the property that Γ_A is commensurable with the Hilbert modular group of K [4].

These matrices A are obviously dense in $Gl_n(\mathbb{R})$.

Now, take a global jet differential of order k, $P \in H^0(\overline{X}, E_{k,m}^{GG} T_{\overline{X}}^*)$. Taking the pull-back, $\varphi_A^* P$ we obtain a k jet differential on a manifold uniformized by a polydisc. Therefore, from example 1.8, we obtain that $\varphi_A(\Delta^n / \Gamma_A) \subset GG(\overline{X})$. By density, we finally get

$$GG(\overline{X}) = \overline{X}.$$

□

4. The isotropic case

The case of Siegel modular varieties is a particular case of the situation where $\mathbb{G}$ is isotropic and $\mathrm{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$.

Theorem 4.1. *If* $\mathrm{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$ *then the Green-Griffiths locus* $GG(\overline{X}) = \overline{X}$.

Proof. Let $\mathbb{G} = SU(D, h)$, where D is a division algebra over $\mathbb{Q}$ and h is a non-degenerate hermitian or skew hermitian form on D^m. $\mathrm{rank}_{\mathbb{Q}}(\mathbb{G})$ coincides with the Witt index of h, i.e. with the dimension of a maximal totally isotropic subspace in D^m see [9]. Let $H_1' =< v >$ be a totally isotropic subspace of dimension 1 on D. Then we can find v' such that $H_1 =< v, v' >$ is a hyperbolic plane i.e. with respect to a properly chosen basis $h_{|H}$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\mathrm{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$ we can find a second hyperbolic plane $H_2 \subset H_1^{\perp}$. Then

$$N \cong SU(D, h_{|H_1}) \times SU(D, h_{|H_2})$$

is a subgroup of G. Therefore we have found a $\mathbb{Q}$-group $N = N_1 \times N_2 \subset \mathbb{G}$. $\Gamma_N := \Gamma \cap N_{\mathbb{Q}}$ is an arithmetic subgroup giving a subvariety

$$\varphi_N : X_{\Gamma_N} = D_N / \Gamma_N \hookrightarrow X_\Gamma := X,$$

whose universal cover D_N is a product $D_1 \times D_2$ corresponding to $N_1(\mathbb{R}) \times N_2(\mathbb{R}) \subset \mathbb{G}_{\mathbb{R}}$.

Now, take a global jet differential of order k, $P \in H^0(\overline{X}, E_{k,m}^{GG} T_{\overline{X}}^*)$. Taking the pull-back, $\varphi_N^* P$ we obtain a k jet differential on a manifold uniformized by a product. Therefore we obtain that $\varphi_N(X_{\Gamma_N}) \subset GG(\overline{X})$.

Consider $g \in \mathbb{G}_{\mathbb{Q}}$ then we have a suvariety

$$\varphi_{gNg^{-1}} : X_{\Gamma_{gNg^{-1}}} = g(D_N)/\Gamma_{gNg^{-1}} \hookrightarrow X_\Gamma := X.$$

Since G is connected $\mathbb{G}_{\mathbb{Q}}$ is dense in $G = \mathbb{G}_{\mathbb{R}}$ (see Theorem 7.7 in [9]). We finally get

$$GG(\overline{X}) = \overline{X}.$$

5. Proof of Theorem 2.1

As the proof of the two previous results made clear, the key point is to find a product of $\mathbb{Q}$-groups in $\mathbb{G}$. If $\mathbb{G} = SU(V, h)$ is of type $U.1$, since G is simple, $\mathbb{G}$ is compact at all but one infinite place where $\mathbb{G}(K_v) = SU(p, q)$. We can diagonalize h and find as above two planes H_1, H_2 where h is of signature $(1, 1)$. The corresponding $\mathbb{Q}$-group is $N_1 = SU(h_{|H_1}) \times SU(h_{|H_2})$ such that $N_1(\mathbb{R}) \cong SU(1, 1) \times SU(1, 1) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Of course, the same reasoning can be applied when $\mathbb{G}$ is of type $O.1$ (with $n \geq 4$) replacing the hermitian form by the symmetric bilinear form. This provides a $\mathbb{Q}$-subgroup N_2 such that $N_2(\mathbb{R}) \cong SO(1, 2) \times SO(1, 2) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Now suppose $\mathbb{G} = SU(V, h)$ is of type $O.2)i)$. Recall that the real rank is equal to $\mathrm{rank}_{\mathbb{R}}(G) = [\frac{n}{2}]$. So our hypothesis on the real rank implies $n \geq 4$. We can therefore find again two planes H_1, H_2 and a $\mathbb{Q}$-subgroup $N_3 = SU(h_{|H_1}) \times SU(h_{|H_2})$, such that $N_3(\mathbb{R}) \cong SO(2, \mathbb{H}) \times SO(2, \mathbb{H}) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ (modulo compact factors $SU(2)$).

Finally if $\mathbb{G} = SU(V, h)$ is of type $S.2$, the hypothesis on the real rank implies $n \geq 2$. So we can find two spaces H_1, H_2 one-dimensional over the quaternion algebra and consider as before the $\mathbb{Q}$-subgroup $N_4 = SU(h_{|H_1}) \times SU(h_{|H_2})$, such that $N_4(\mathbb{R}) = Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R}) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

□

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